EQUIDISTRIBUTION OF SMALL POINTS, RATIONAL DYNAMICS, AND POTENTIAL THEORY

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ABSTRACT. Given a dynamical system associated to a rational function $\varphi(T)$ on $\mathbb{P}^1$ of degree at least 2 with coefficients in a number field k, we show that for each place v of k, there is a unique probability measure $\mu_{\varphi,v}$ on the Berkovich space $\mathbb{P}^1_{\mathrm{Berk},v}/\mathbb{C}_v$ such that if $\{z_n\}$ is a sequence of points in $\mathbb{P}^1(\overline{k})$ whose φ-canonical heights tend to zero, then the z_n's and their Galois conjugates are equidistributed with respect to $\mu_{\varphi,v}$. In the archimedean case, $\mu_{\varphi,v}$ coincides with the well-known canonical measure associated to φ. This theorem generalizes a result of Baker-Hsia [BH] when $\varphi(z)$ is a polynomial.

The proof uses a polynomial lift $F(x,y) = (F_1(x,y), F_2(x,y))$ of φ to construct a two-variable Arakelov-Green's function $g_{\varphi,v}(x,y)$ for each v. The measure $\mu_{\varphi,v}$ is obtained by taking the Berkovich space Laplacian of $g_{\varphi,v}(x,y)$, using a theory developed in [RB]. The other ingredients in the proof are (i) a potential-theoretic energy minimization principle which says that $\iint g_{\varphi,v}(x,y)\,d\nu(x)d\nu(y)$ is uniquely minimized over all probability measures ν on $\mathbb{P}^1_{\mathrm{Berk},v}$ when $\nu = \mu_{\varphi,v}$, and (ii) a formula for *homogeneous transfinite diameter* of the v-adic filled Julia set $K_{F,v} \subset \mathbb{C}_v^2$ in terms of the resultant $\mathrm{Res}(F)$ of F_1 and F_2. The resultant formula, which generalizes a formula of DeMarco [DeM], is proved using results from [RLV] about Chinburg's *sectional capacity*. A consequence of the resultant formula is that the product of the homogeneous transfinite diameters over all places is 1.

Let k be a number field, and let $\varphi(T) \in k(T)$ be a rational function of degree $d \geq 2$. In this paper we investigate the equidistribution properties of small points relative to the canonical dynamic height $\hat{h}_\varphi(z)$. We show that for each place v of k, there is a probability measure $\mu_{\varphi,v}$ such that if $\{z_n\}$ is a sequence of distinct points in $\mathbb{P}^1(\overline{k})$ satisfying $\hat{h}_\varphi(z_n) \to 0$, then the Galois conjugates of the z_n (regarded as embedded in $\mathbb{P}^1(\mathbb{C}_v)$) are equidistributed relative to $\mu_{\varphi,v}$. More precisely, if δ_n is the discrete probability measure supported equally on the conjugates of z_n, then the sequence δ_n converges weakly to $\mu_{\varphi,v}$ for each v. When v is archimedean, $\mu_{\varphi,v}$ is the well-known canonical measure on $\mathbb{P}^1(\mathbb{C})$ supported on the Julia set of φ which was constructed by Lyubich [Ly] and Freire-Lopes-Mañé [FLM]. (See [Mi, §4] for the definition of the Julia set of a rational map.)

When v is nonarchimedean, $\mu_{\varphi,v}$ is a measure on the Berkovich space $\mathbb{P}^1_{\mathrm{Berk},v}$ over $\mathbb{C}_v$ constructed by the authors in [RB]. It has the same invariance properties relative to φ as the canonical measure in the archimedean case.

Conceptually, the proof is very simple. Its main ingredients are an energy-minimization principle, established at each place v for the Arakelov Green's function $g_{\varphi,v}(x,y)$ assocated to $\mu_{\varphi,v}$, and two global inequalities, an upper bound coming from the assumption that $\hat{h}_\varphi(z_n) \to$

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0, and a lower bound coming from the product formula. Combining these ingredients yields equidistribution simultaneously at all places v.

At nonarchimedean places, the Arakelov Green's function $g_{\varphi,v}(x,y)$ is new. We construct it by lifting φ to a polynomial map $F : \mathbb{C}_v^2 \to \mathbb{C}_v^2$, and using the homogeneous local height associated to the filled Julia set $K_{F,v}$ of this lift. This approach was inspired by DeMarco [DeM], who introduced the *homogeneous capacity* $c^0(K)$ for sets $K \subset \mathbb{C}^2$, and proved for archimedean v that

$$c^0(K_{F,v}) \;=\; |\mathrm{Res}(F)|^{-1/d(d-1)} \; .$$

Although the homogeneous capacity does not easily generalize to nonarchimedean places, for arbitrary v we introduce the closely related *homogeneous transfinite diameter $d_\infty^0(K_v)$* for sets $K_v \subset \mathbb{C}_v^2$. We generalize DeMarco's formula by showing that

$$d_\infty^0(K_{F,v}) \;=\; |\mathrm{Res}(F)|_v^{-1/d(d-1)} \tag{0.1}$$

for each v. This is proved by relating the homogeneous transfinite diameter to the *sectional capacity* studied in [Chi] and [RLV]. The fact that $\prod_v d_\infty^0(K_{F,v}) = 1$, which follows from the product formula applied to (0.1), is the key to the global lower bound mentioned above.

A philosophical idea which we hope to promote is the use of Berkovich spaces as a natural setting for nonarchimedean Arakelov theory and equidistribution theorems. This point of view has been most strongly espoused by A. Chambert-Loir [ChL]. The foundational results concerning potential theory on the Berkovich projective line which are used in this paper can be found in [RB]. Many of these results are proved for Berkovich curves of arbitrary genus in the doctoral thesis of A. Thuillier [Th], a recent student of Chambert-Loir.

P. Autissier [Au] has proved the archimedean part of the dynamical equidistribution theorem using Arakelov-theoretic methods. A proof of the nonarchimedean (Berkovich space) part of the theorem, also based on ideas from Arakelov theory, has been announced by Chambert-Loir [ChL].

C. Favre and J. Rivera-Letelier have also announced a proof of the dynamical equidistribution theorem. Their preprints [FRL1], [FRL2] give a proof of Theorem 2.3 together with another construction of the canonical measure on $\mathbb{P}^1_{\mathrm{Berk},v}$ attached to a rational map φ. They also prove a Berkovich space analogue of Theorem 2.1 below. The technical foundation for their work can be found in the monograph by Favre and Jonsson [FJ], in Rivera-Letelier's thesis [R-L1], and in a manuscript of Rivera-Letelier [R-L2]. As with our approach, Favre and Rivera-Letelier's proof of of Theorem 2.3 is ultimately based on the product formula and an adelic energy-minimization theorem. However, there are also a number of differences between the two proofs.

Finally we note that in the nonarchimedean case, the construction of J. Piniero, L. Szpiro and T. Tucker [PST], which works scheme-theoretically with blowups of models of $\mathbb{P}^1/\mathrm{Spec}(\mathcal{O}_v)$ attached to iterates of $\varphi(T)$, yields a sequence of discrete measures which can be shown to converge to the canonical measure on $\mathbb{P}^1_{\mathrm{Berk},v}$.

1. NOTATION

We set the following notation and normalizations, which will be used throughout the paper unless otherwise noted.

k a number field.

$\mathcal{O}_k$ the ring of integers of k.

M_k the set of places of k.

k_v the completion of k at v.

$\mathcal{O}_v$ the ring of integers in k_v.

q_v the order of the residue field of k_v. If v is archimedean, we put $q_v = e$ if $k_v \cong \mathbb{R}$, and $q_v = e^2$ if $k_v \cong \mathbb{C}$.

$\mathbb{C}_v$ the completion of a fixed algebraic closure $\overline{k}_v$ of k_v. Throughout the paper, we fix a choice of an embedding of $\overline{k}$ into $\mathbb{C}_v$ for each $v \in M_k$ (though all of our conclusions will be independent of the choices made). If v is nonarchimedean, we write $\widehat{\mathcal{O}}_v$ for the ring of integers of $\mathbb{C}_v$.

$|x|_v$ the canonical absolute value on k_v given by the modulus of additive Haar measure. If $|x|_v'$ is the unique absolute value on k in the equivalence class of $v \in M_k$ that extends the standard absolute value on the completion $\mathbb{Q}_v$, then $|x|_v = (|x|_v')^{[k_v:\mathbb{Q}_v]}$. With this normalization, the product formula holds in the form $\prod_v |\alpha|_v = 1$ for each $\alpha \neq 0$ in k. Each $|x|_v$ extends uniquely to an absolute value on $\mathbb{C}_v$, the completion of the algebraic closure of k_v.

h the absolute logarithmic Weil height $h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}$, defined for $[x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$ by

$$h\big([x_0 : \cdots : x_n]\big) = \frac{1}{[k : \mathbb{Q}]} \sum_{v \in M_k} \log \max\{|x_0|_v, \ldots, |x_n|_v\}.$$

φ A rational function on $\mathbb{P}^1$ defined over k.

2. Overview

2.1. **An equidistribution result for rational functions on $\mathbb{P}^1$.** Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a finite morphism (i.e., a nonconstant rational function) of degree $d \geq 2$ defined over the number field k. Iterating φ gives rise to a dynamical system on $\mathbb{P}^1(\mathbb{C}_v)$ for all places v of k. When v is archimedean, this type of dynamical system has been extensively studied since the pioneering work of Fatou and Julia in the early 20th century. Just as one defines the Néron-Tate canonical height on an elliptic curve by iteration, one can define the *dynamical height*

$$\hat{h}_\varphi : \mathbb{P}^1(\overline{k}) \to \mathbb{R}$$

attached to the rational function φ by the rule

$$\hat{h}_\varphi(z) \;=\; \lim_{n \to \infty} \frac{1}{d^n} h(\varphi^{(n)}(z)).$$

Here $\varphi^{(n)}$ denotes the n-fold iterate $\varphi \circ \cdots \circ \varphi$. By a general result of Call and Silverman [CS], the hypothesis $d \geq 2$ guarantees that the above limit exists.

The dynamical height $\hat{h}_\varphi$ is uniquely characterized by the following two properties:

(1) The difference $|\hat{h}_\varphi - h|$ is bounded.

(2) $\hat{h}_\varphi \circ \varphi = d \cdot \hat{h}_\varphi$.

It follows from [CS] that $\hat{h}_\varphi(z) \geq 0$ for all $z \in \mathbb{P}^1(\overline{k})$, and $\hat{h}_\varphi(z) = 0$ if and only if z is *preperiodic* for φ, meaning that the orbit $\{\varphi^{(n)}(z) : n \in \mathbb{N}\}$ of z under iteration of φ is a finite set. Additionally, we have $\hat{h}_\varphi(\sigma z) = \hat{h}_\varphi(z)$ for all $z \in \mathbb{P}^1(\overline{k})$ and all $\sigma \in \mathrm{Gal}(\overline{k}/k)$.

If $\varphi(z) = z^2$, then $\hat{h}_\varphi$ is the usual logarithmic Weil height h on $\mathbb{P}^1(\overline{\mathbb{Q}})$. Another well-known height which can be defined by dynamical methods is the Néron-Tate canonical height on an elliptic curve. If k is a number field and E/k is an elliptic curve with Weierstrass equation $y^2 = f(x)$, let φ be the degree 4 rational function on $\mathbb{P}^1$ given by $x \circ [2]$. Then for $P \in E(\overline{k})$ we have $\hat{h}(P) = \hat{h}_\varphi(x(P))$.

For any rational function φ on $\mathbb{P}^1$ of degree $d \geq 2$ defined over $\mathbb{C}$, Lyubich [Ly], and independently Freire, Lopes, and Mañé [FLM], constructed a natural probability measure μ_φ attached to the dynamical system $\{\varphi^{(n)} : n \in \mathbb{N}\}$. We will refer to the measure μ_φ as the *canonical measure* attached to φ. In order to characterize μ_φ, we recall the following definition. A point $z_0 \in \mathbb{P}^1(\mathbb{C})$ is said to be *exceptional* if the set $\{\varphi^{(-n)}(z_0) : n \in \mathbb{N}\}$ of backward iterates of z_0 is finite. It is known (see [Mi]) that there are at most 2 exceptional points for φ in $\mathbb{P}^1(\mathbb{C})$. Proofs of the following theorem can be found in [Ly], [FLM], and [HP].

Theorem 2.1. *There exists a probability measure μ_φ (independent of z_0) such that:*
   A) For any non-exceptional point $z_0 \in \mathbb{P}^1(\mathbb{C})$, let δ_n be the probability measure

$$\frac{1}{d^n} \sum_{\varphi^{(n)}(z)=z_0} \delta_z,$$

where the points in the sum are counted with multiplicities and δ_z denotes the Dirac measure on $\mathbb{P}^1(\mathbb{C})$ giving mass 1 to the point z. Then the sequence of measures δ_n converges weakly to μ_φ.
   B) μ_φ is the unique measure on $\mathbb{P}^1(\mathbb{C})$ with no point masses such that $\varphi^*(\mu_\varphi) = d \cdot \mu_\varphi$ as $(1,1)$-currents.

When φ is a polynomial, Theorem 2.1 was originally proved by Brolin, and the measure μ_φ is known in that case as *Brolin's measure*. Brolin's measure coincides with the equilibrium measure (in the sense of potential theory) on the Julia set of φ.

We will now briefly recall the construction of the Berkovich space $\mathbb{P}^1_{\mathrm{Berk},v}$ associated to the projective line over $\mathbb{C}_v$, where v is a nonarchimedean place of k.

The Berkovich unit disc $\mathcal{B}(0,1)$ is the set of all continuous multiplicative seminorms on the Tate algebra $\mathbb{C}_v\langle T \rangle$ (see [Ber, §1.4], [RB, §1]). Examples of elements of $\mathcal{B}(0,1)$ include the evaluation seminorms $[f]_a = |f(a)|_v$ for $a \in \mathbb{C}_v$ with $|a|_v \leq 1$; sup norms $[f]_{B(a,r)} = \sup_{z \in B(a,r)} |f(z)|_v$ for discs $B(a,r) = \{z \in \mathbb{C}_v : |z - a|_v \leq r\}$; and limit norms associated to nested sequences of discs $B(a_1, r_1) \supset B(a_2, r_2) \supset \cdots$, defined by

$$[f]_x = \lim_{i \to \infty} [f]_{B(a_i, r_i)}.$$

A theorem of Berkovich says that all continuous multiplicative seminorms on $\mathbb{C}_v\langle T \rangle$ arise in this way. Following Chambert-Loir [ChL], we call the point $\zeta_0 \in \mathcal{B}(0,1)$ corresponding to the Gauss norm $\|f\| = [f]_{B(0,1)}$ the *Gauss point*. Given a point $x \in \mathcal{B}(0,1)$ corresponding either to a disc $B(a,r)$ or to a point $a = B(a,0)$ (which can be thought of as a degenerate disc), there is a path $\{[\ ]_{B(a,t)} : r \leq t \leq 1\}$ connecting x to the Gauss point. Given a collection of discs, the union of the corresponding paths forms a subtree of $\mathcal{B}(0,1)$ rooted at ζ_0. From this, one sees that $\mathcal{B}(0,1)$ is an infinitely branched real tree, with countably many branches emanating from each point corresponding to a disc with radius $r \in |\mathbb{C}_v^\times|_v$.

As a set, the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk},v}$ over $\mathbb{C}_v$ is obtained by gluing together two copies of $\mathcal{B}(0,1)$. It is made into a topological space by equipping it with the *Gelfand topology*, the weakest topology such that each set of the form

$$U_{a,b}(f) = \{x \in \mathbb{P}^1_{\mathrm{Berk},v} : a < [f]_x < b\}$$

for $a, b \in \mathbb{R}$ and $f \in \mathbb{C}_v(T)$ is open. The space $\mathbb{P}^1_{\mathrm{Berk},v}$ is also equipped with a sheaf of rings $\mathcal{O}_X$, constructed using localizations of Tate algebras; see [Ber] for details. There is a natural inclusion $\mathbb{P}^1(\mathbb{C}_v) \subset \mathbb{P}^1_{\mathrm{Berk},v}$ (which associates to a point of $\mathbb{P}^1(\mathbb{C}_v)$ the corresponding "evaluation seminorm") that induces the usual (ultrametric) topology on $\mathbb{P}^1(\mathbb{C}_v)$, and $\mathbb{P}^1(\mathbb{C}_v)$ is dense in $\mathbb{P}^1_{\mathrm{Berk},v}$ under this inclusion.

If $\varphi(T) \in \mathbb{C}_v(T)$ is a nonconstant rational function, then φ acts on $\mathbb{P}^1_{\mathrm{Berk}}$ by

$$[f]_{\varphi(x)} = [f \circ \varphi]_x$$

for all $f \in \mathbb{C}_v(T)$. This coincides with the usual action of φ on $\mathbb{P}^1(\mathbb{C}_v) \subset \mathbb{P}^1_{\mathrm{Berk},v}$.

As a topological space, $\mathbb{P}^1_{\mathrm{Berk},v}$ is compact, Hausdorff, and path-connected, in contrast with $\mathbb{P}^1(\mathbb{C}_v)$, which is completely disconnected and not even locally compact. Thus $\mathbb{P}^1_{\mathrm{Berk},v}$ is a much more suitable space for doing measure theory and potential theory than $\mathbb{P}^1(\mathbb{C}_v)$. The space $\mathbb{P}^1_{\mathrm{Berk},v}$ is also metrizable, although there is not a canonical metric on it.

Remark 2.2. If v is archimedean, one can define $\mathbb{P}^1_{\mathrm{Berk},v}$ over $\mathbb{C}$ in a similar way using continuous multiplicative seminorms on $\mathbb{C}\langle T \rangle$. By the Gelfand-Mazur theorem, every such seminorm arises from evaluation at a point. Thus $\mathbb{P}^1_{\mathrm{Berk},v}/\mathbb{C}$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$.

In Theorem 2.1, note that if $z_n \in \varphi^{(-n)}(z_0)$, then

$$\hat{h}_\varphi(z_n) = \frac{1}{d^n}\hat{h}_\varphi(z_0) \to 0$$

as $n \to \infty$. Also, note that if φ is defined over the number field k and $z_0 \in k$, then the set $\varphi^{(-n)}(z_0)$ is stable under $\mathrm{Gal}(\overline{k}/k)$. We will prove the following adelic equidistribution theorem, motivated by Theorem 2.1 and by the archimedean equidistribution theorems of Bilu [Bi] and Szpiro-Ullmo-Zhang [SUZ].

Theorem 2.3 (Main Theorem). *For each place $v \in M_k$, there exists a canonical probability measure $\mu_{\varphi,v}$ on the Berkovich space $\mathbb{P}^1_{\mathrm{Berk},v}/\mathbb{C}_v$ such that the following holds: Suppose z_n is a sequence of distinct points of $\mathbb{P}^1(\overline{k})$ with $\hat{h}_\varphi(z_n) \to 0$. For $v \in M_k$, let δ_n be the discrete probability measure on the Berkovich space $\mathbb{P}^1_{\mathrm{Berk},v}/\mathbb{C}_v$ supported equally on the Galois conjugates of z_n. Then the sequence of measures δ_n converges weakly to $\mu_{\varphi,v}$ for all $v \in M_k$.*

When φ is a polynomial, the archimedean part of Theorem 2.3 was proved by Baker-Hsia in [BH]. The present paper provides a conceptual simplification of their method, and applies to arbitrary rational functions. We note that the case of a rational function is more difficult than the polynomial case, due to the absence of a fixed pole at infinity. A weaker version of the nonarchimedean part of Theorem 2.3, formulated in terms of "pseudo-equidistribution", was also proved for the polynomial case in Baker-Hsia in [BH]. Here we clarify the meaning of pseudo-equidistribution by using Arakelov Green's functions and Berkovich spaces.

When $\varphi(z) = z^2$, the archimedean part of Theorem 2.3 specializes to (and was motivated by) the following well-known result of Bilu:

Theorem 2.4 (Bilu [Bi])**.** *Let z_n be a sequence of distinct points in $\mathbb{P}^1(\overline{\mathbb{Q}})$, and suppose that $h(z_n) \to 0$. Let δ_n be the discrete probability measure on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ which is supported with equal mass at each Galois conjugate of z_n. Then the sequence of measures δ_n converges weakly to the uniform probability measure μ_{S^1} on the unit circle $\{|z| = 1\}$.*

For previous explorations of the relationship between Bilu's theorem and potential theory, see [Bo] and [R3].

3. ADELIC DYNAMICS ON $\mathbb{P}^1$

3.1. **Dynamical heights associated to rational functions.** Recall that $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational function of degree $d \geq 2$ defined over a number field k.

The map φ can be represented in homogeneous coordinates as

$$\varphi([z_0 : z_1]) \;=\; [F_1(z_0, z_1) : F_2(z_0, z_1)]$$

for some homogeneous polynomials $F_1, F_2 \in k[x, y]$ of degree d with no common linear factor over $\overline{k}$. (Note that since F_1 and F_2 factor into linear terms over $\overline{k}$, F_1 and F_2 have a common factor over $\overline{k}$ if and only if they have a common linear factor over $\overline{k}$.) The polynomials F_1, F_2 are uniquely determined by φ up to multiplication by a common scalar $c \in k^*$.

Dehomogenizing by setting $z = z_1/z_0$, we obtain

$$\varphi(z) \;=\; \frac{f_2(z)}{f_1(z)}$$

with $f_i \in k[z]$ and $\max\{\deg(f_1), \deg(f_2)\} = d$.

We will often want to work with the degree d homogeneous polynomials F_1 and F_2, so we now fix a choice of $F_1, F_2 \in k[X, Y]$. This allows us to consider the mapping

$$F = (F_1, F_2) : \mathbb{A}^2(\overline{k}) \to \mathbb{A}^2(\overline{k})$$

as a global lifting of φ.

Let $\mathrm{Res}(F) := \mathrm{Res}(F_1, F_2)$ denote the homogeneous resultant of the polynomials F_1 and F_2 (see e.g. [DeM, §6]). Since F_1 and F_2 have no common linear factor over $\overline{k}$, we have $\mathrm{Res}(F) \neq 0$, and $F(z_0, z_1) = (0, 0)$ if and only if $(z_0, z_1) = (0, 0)$.

In the archimedean case, define $\|(z_0, z_1)\|_v = \sqrt{|z_0|^2 + |z_1|^2}^{[k_v:\mathbb{R}]}$; in the nonarchimedean case, put $\|(z_0, z_1)\|_v := \max\{|z_0|_v, |z_1|_v\}$. We begin with the following simple lemma.

Lemma 3.1. *For each place v of k, there are constants $0 < C_v \leq D_v$ such that for all $z \in \mathbb{C}_v^2$,*

$$C_v(\|z\|_v)^d \;\leq\; \|F(z)\|_v \;\leq\; D_v(\|z\|_v)^d \;. \tag{3.2}$$

For all but finitely many v, we may take $C_v = D_v = 1$.

Proof: First suppose v is archimedean. Identify $\mathbb{C}_v$ with $\mathbb{C}$. Since $\partial B_v(1) = \{(x, y) \in \mathbb{C}^2 : \max(|x|, |y|) = 1\}$ is compact, and since the only common zero of $F_1(z)$ and $F_2(z)$ is the origin, the constants

$$C_v = \min_{z \in \partial B_v(1)} \|F(z)\|_v \;, \quad D_v = \max_{z \in \partial B_v(1)} \|F(z)\|_v$$

satisfy $0 < C_v \leq D_v$. By homogeneity, (3.2) holds for all $z \in \mathbb{C}^2$.

Now let v be nonarchimedean. Write

$$\partial B_v(1) = \{(x,y) \in \mathbb{C}_v^2 : \max(|x|_v, |y|_v) = 1\} .$$

Let $C_v' := |\text{Res}(F)|_v$, and let D_v be an upper bound for the absolute values of the coefficients of F_1 and F_2.

By a well-known property of the resultant of two homogeneous polynomials of degree d [VdW, §5.8], there exist polynomials $g_1(x,y)$, $g_2(x,y)$, and $h_1(x,y)$, $h_2(x,y)$, which are homogeneous of degree $d-1$ in x and y and whose coefficients lie in k, such that

$$\begin{aligned}
g_1(x,y)F_1(x,y) + g_2(x,y)F_2(x,y) &= \text{Res}(F)x^{2d-1} , \\
h_1(x,y)F_1(x,y) + h_2(x,y)F_2(x,y) &= \text{Res}(F)y^{2d-1} .
\end{aligned}$$

For each $(x,y) \in \partial B_v(1)$, all of $|g_1(x,y)|_v$, $|g_2(x,y)|_v$, $|h_1(x,y)|_v$ and $|h_2(x,y)|_v$ are $\leq C_v''$ for some constant $C_v'' > 0$ independent of (x,y). Furthermore, we may take $C_v'' = 1$ for almost all v. By the ultrametric inequality,

$$\begin{aligned}
C_v'|x|_v^{2d-1} &\leq C_v'' \max(|F_1(x,y)|_v, |F_2(x,y)|_v) , \\
C_v'|y|_v^{2d-1} &\leq C_v'' \max(|F_1(x,y)|_v, |F_2(x,y)|_v) .
\end{aligned}$$

Put $C_v := C_v'/C_v''$. Then for each $z = (x,y) \in \partial B_v(1)$,

$$C_v \cdot \max(|x|_v, |y|_v)^{2d-1} \leq \|F(z)\|_v .$$

However, if $z \in \partial B_v(1)$ then $\max(|x|_v, |y|_v) = \|z\|_v = 1$, so $C_v \leq \|F(z)\|_v$ for all $z \in \partial B_v(1)$. The first inequality in (3.2) follows by homogeneity. The second follows trivially by the ultrametric inequality.

Finally, since the resultant and the coefficients of the F_i are elements of k, independent of v, we can take $C_v = D_v = 1$ for all but finitely many v. □

Remark 3.3. If v is nonarchimedean and F_1, F_2 have v-integral coefficients, then we may choose $g_1(x,y)$, $g_2(x,y)$, and $h_1(x,y)$, $h_2(x,y)$ to have v-integral coefficients as well. In this case we can take $C_v = |\text{Res}(F)|_v$ and $D_v = 1$.

Corollary 3.4. *For each v, there are radii $0 < r_v \leq R_v$ such that for each $z \in \mathbb{C}_v^2$ with $\|z\|_v \leq r_v$, we have $\|F(z)\|_v \leq \|z\|_v \cdot (\|z\|_v/r_v)^{d-1}$, and for each z with $\|z\|_v \geq R_v$, we have $\|F(z)\|_v \geq \|z\|_v \cdot (\|z\|_v/R_v)^{d-1}$. In particular,*

$$\begin{aligned}
F(B_v(r_v)) &\subseteq B_v(r_v) , \\
F(\mathbb{C}_v^2 \backslash B_v(R_v)) &\subseteq \mathbb{C}_v^2 \backslash B_v(R_v) .
\end{aligned}$$

For all but finitely many v, we can take $r_v = R_v = 1$.

Proof: Let C_v and D_v be as in Lemma 3.1, and put $r_v = D_v^{-1/(d-1)}$, $R_v = C_v^{-1/(d-1)}$. If $\|z\|_v \leq r_v$, then

$$\begin{aligned}
\|F(z)\|_v &\leq D_v\|z\|_v^d = r_v^{-(d-1)}\|z\|_v^d \\
&= \|z\|_v \cdot (\|z\|_v/r_v)^{d-1} .
\end{aligned}$$

Similarly if $\|z\|_v \geq R_v$, then

$$\begin{aligned}
\|F(z)\|_v &\geq C_v\|z\|_v^d = R_v^{-(d-1)}\|z\|_v^d \\
&= \|z\|_v \cdot (\|z\|_v/R_v)^{d-1} .
\end{aligned}$$

For each v with $C_v = D_v = 1$, we have $r_v = R_v = 1$. □

Recall that the global dynamical height $\hat{h}_\varphi : \mathbb{P}^1(\overline{k}) \to \mathbb{R}$ is defined by

$$\hat{h}_\varphi(z) = \lim_{n \to \infty} \frac{1}{d^n} h(\varphi^{(n)}(z)).$$

The choice of a global lifting F of φ allows us to decompose the global dynamical height into a sum of local heights as follows.

For $v \in M_k$ and $z = (z_0, z_1) \in \mathbb{C}_v^2 \backslash \{0\}$, define the *homogeneous local dynamical height* $\hat{H}_{F,v} : \mathbb{C}_v^2 \backslash \{0\} \to \mathbb{R}$ by

$$\hat{H}_{F,v}(z) := \lim_{n \to \infty} \frac{1}{d^n} \log \|F^{(n)}(z)\|_v.$$

By convention, we define $\hat{H}_{F,v}(0,0) := -\infty$.

Lemma 3.5. *The limit $\lim_{n \to \infty} \frac{1}{d^n} \log \|F^{(n)}(z)\|_v$ exists for all $z \in \mathbb{C}_v^2 \backslash \{0\}$, and $\frac{1}{d^n} \log \|F^{(n)}(z)\|_v$ converges uniformly on $\mathbb{C}_v^2 \backslash \{0\}$ to $\hat{H}_{F,v}(z)$.*

Proof: The proof is by a standard telescoping series argument (see e.g. [HP]). Define

$$\begin{aligned} T_j(z) &:= \frac{1}{d^{j+1}} \log \|F^{(j+1)}(z)\|_v - \frac{1}{d^j} \log \|F^{(j)}(z)\|_v \\ &= \frac{1}{d^j} \left(\frac{1}{d} \log \|F^{(j+1)}(z)\|_v - \log \|F^{(j)}(z)\|_v \right). \end{aligned}$$

By (3.2) we see that

$$C_v(\|F^{(j)}(z)\|_v)^d \ \leq \ \|F^{(j+1)}(z)\|_v \ \leq \ D_v(\|F^{(j)}(z)\|_v)^d$$

for all $z \in \mathbb{C}_v^2$. Applying these inequalities to the sequence T_j yields the estimate

$$|T_j(z)| \ \leq \ \frac{C}{d^{j+1}},$$

where $C := \max\{\log D_v, -\log C_v\}$.

It follows easily that $H_n := \sum_{j=0}^n T_j$ is a Cauchy sequence, and that the series defining $\hat{H}_{F,v}(z)$ converges uniformly on $\mathbb{C}_v^2 \backslash \{0\}$. $\square$

Remark 3.6. The proof yields the explicit bound

$$|\hat{H}_{F,v}(z) - \log \|z\|_v| \ \leq \ \frac{C}{d-1},$$

valid for all $z \in \mathbb{C}_v^2 \backslash \{0\}$.

Note that by the definitions of the local and global canonical heights, if $x \in \mathbb{P}^1(k)$, then for any representation $x = [x_0 : x_1]$ with $x_0, x_1 \in k$, we have

$$\hat{h}_\varphi(x) = \frac{1}{[k : \mathbb{Q}]} \sum_{v \in M_k} \hat{H}_{F,v}(x_0, x_1).$$

By the product formula, the right side is independent of the choice of lifting.

Also, note that the definition of $\hat{H}_{F,v}$ is independent of the norm used to define it. This follows easily from the equivalence of norms on $\mathbb{C}_v^2$.

The homogeneous local dynamical height $\hat{H}_{F,v}$ has the following properties, and in fact is uniquely characterized by them:

(LH1) The difference $|\hat{H}_{F,v}(z) - \log \|z\|_v|$ is bounded.

(LH2) $\hat{H}_{F,v}(F(z)) = d \cdot \hat{H}_{F,v}(z)$.

(LH3) $\hat{H}_{F,v}$ *scales logarithmically*, i.e., for all $c \in \mathbb{C}_v^*$,

$$\hat{H}_{F,v}(cz) = \hat{H}_{F,v}(z) + \log|c|_v .$$

3.2. **The filled Julia set.** By definition, the *filled Julia set* $K_{F,v}$ of F in $\mathbb{C}_v^2$ is the set of all $z \in \mathbb{C}_v^2$ for which the iterates $F^{(n)}(z)$ remain bounded. Clearly $F^{-1}(K_{F,v}) = K_{F,v}$, and the same is true for each $F^{(-n)}$. Since all norms on $\mathbb{C}_v^2$ are equivalent, the set $K_{F,v}$ is independent of which norm is used to define it.

By Corollary 3.4, we have $B_v(r_v) \subseteq K_{F,v}$, so $K_{F,v}$ cannot be too small. Moreover:

Lemma 3.7. *With R_v as in Corollary 3.4, we have*

$$F^{(-1)}(B_v(R_v)) \supseteq F^{(-2)}(B_v(R_v)) \supseteq \cdots$$

and

$$K_{F,v} = \cap_{n=1}^\infty F^{(-n)}(B_v(R_v)).$$

Proof: The assertion that $F^{(-n)}(B_v(R_v)) \supseteq F^{(-n-1)}(B_v(R_v))$ for all $n \geq 1$ is equivalent to the statement that if $F^{(n)}(z) \notin B_v(R_v)$ then $F^{(n+1)}(z) \notin B_v(R_v)$. This follows from Corollary 3.4, which shows that $F(\mathbb{C}_v^2 \backslash B_v(R_v)) \subseteq \mathbb{C}_v^2 \backslash B_v(R_v)$.

It also follows from Corollary 3.4 that if $\|z\|_v > R_v$, then

$$\lim_{n\to\infty} \|F^{(n)}(z)\|_v = \infty .$$

Thus, $K_{F,v} \subset B_v(R_v)$. By iteration, $K_{F,v} \subset F^{(-n)}(B_v(R_v))$ for each n.

The fact that $K_{F,v} = \cap_{n=1}^\infty F^{(-n)}(B_v(R_v))$ now follows, since if $z \notin K_{F,v}$, there is some n for which $\|F^{(n)}(z)\|_v > R_v$, and so $z \notin F^{(-n)}(B_v(R_v))$. $\qquad\square$

The filled Julia set $K_{F,v}$ can be thought of as the 'unit ball' with respect to the dynamical local height $\hat{H}_{F,v}$:

Lemma 3.8. *For each place v of k,*

$$K_{F,v} = \{z \in \mathbb{C}_v^2 : \hat{H}_{F,v}(z) \leq 0\}.$$

Proof: If $z \in K_{F,v}$ then there exists $M > 0$ such that $\|F^{(n)}(z)\|_v \leq M$ for all n, and therefore $\hat{H}_{F,v}(z) \leq \lim_{n\to\infty} \frac{1}{d^n} \log M = 0$.

Conversely, suppose $z \notin K_{F,v}$. Then for n_0 sufficiently large, $\beta := \|F^{(n_0)}(z)\|_v > R_v$. Let $\alpha := \beta/R_v > 1$. Then by Corollary 3.4 and induction on n, it follows that

$$\|F^{(n+n_0)}(z)\|_v > \beta \cdot \alpha^{d^n - 1}$$

for all $n \geq 0$.

Therefore

$$\hat{H}_{F,v}(z) \geq \lim_{n\to\infty} \frac{1}{d^{n+n_0}} \left((d^n - 1)\log\alpha + \log\beta\right) = \frac{1}{d^{n_0}}\log\alpha > 0.$$

$\qquad\square$

In general, it is difficult to describe $K_{F,v}$ explicitly. However, the following lemma shows that it is 'trivial' for all but finitely many v.

Lemma 3.9. *Suppose $F_1, F_2 \in \mathcal{O}_k[x, y]$. If v is a nonarchimedean place of k such that $|\mathrm{Res}(F)|_v = 1$, then $K_{F,v} = B(0,1)^2$ is the unit polydisk in $\mathbb{C}_v^2$ and $\hat{H}_{F,v}(z) = \log \|z\|_v$ for all $z \in \mathbb{C}_v^2$.*

Proof: By Remark 3.3, it follows that $\|F(z)\|_v = \|z\|_v^d$, and therefore $\|F^{(n)}(z)\|_v = \|z\|_v^{d^n}$ for all $n \geq 1$. The result follows immediately. $\qquad\qquad\qquad\qquad\qquad\square$

3.3. The homogeneous transfinite diameter.

Let v be a place of k, and let $K \subset \mathbb{C}_v^2$ be a nonempty bounded set. For $z = (z_0, z_1), w = (w_0, w_1) \in \mathbb{C}_v^2$, put

$$z \wedge w := z_0 w_1 - z_1 w_0 .$$

By analogy with the classical transfinite diameter, for $n \geq 2$ we let

$$d_n^0(K) := \sup_{z_1,\dots,z_n \in K} \left(\prod_{i \neq j} |z_i \wedge z_j|_v \right)^{\frac{1}{n(n-1)}} .$$

Lemma 3.10. *The sequence of nonnegative real numbers $d_n^0(K)$ is non-increasing. In particular, the quantity $d_\infty^0(K) := \lim_{n \to \infty} d_n^0(K)$ is well-defined.*

Proof: We claim that $d_n^0(K) \geq d_{n+1}^0(K)$ for all n. The proof is the same as for the classical transfinite diameter: write $P_n(z_1, \dots, z_n) = \prod_{i \neq j} (z_i \wedge z_j)$, take $\varepsilon > 0$, and choose $w_1, \dots, w_{n+1} \in K$ with $|P_{n+1}(w_1, \dots, w_{n+1})|_v \geq (d_{n+1}^0(K) - \varepsilon)^{n(n+1)}$. For each $\ell = 1, \dots, n+1$ write $\hat{w}_\ell = (w_1, \dots, w_{\ell-1}, w_{\ell+1}, \dots, w_{n+1})$. By definition, $d_n^0(K)^{n(n-1)} \geq |P_n(\hat{w}_\ell)|_v$ for each ℓ. It follows that

$$\begin{aligned} d_n^0(K)^{(n+1)n(n-1)} &\geq \prod_{\ell=1}^{n+1} |P_n(\hat{w}_\ell)|_v = \left(\prod_{i \neq j}^{n+1} |w_i \wedge w_j|_v \right)^{n-1} \\ &\geq (d_{n+1}^0(K) - \varepsilon)^{(n+1)n(n-1)} . \end{aligned}$$

This holds for each $\varepsilon > 0$, so $d_n^0(K) \geq d_{n+1}^0(K)$. $\qquad\qquad\qquad\qquad\square$

We call $d_\infty^0(K)$ the *homogeneous transfinite diameter* of K.

We will now show that when v is archimedean and $K \subset \mathbb{C}^2$ is compact, $d_\infty^0(K)$ coincides with the *homogeneous capacity* introduced by DeMarco [DeM]. We recall the definition.

Definition 3.11. If $K \subseteq \mathbb{C}^2$ is compact and nonempty, define $c^0(K)$ by

$$-\log c^0(K) := \inf_{\nu \in \mathbb{P}(K)} I(\nu),$$

where $\mathbb{P}(K)$ is the space of probability measures supported on K, and

$$I(\nu) := -\iint_{K \times K} \log |z \wedge w| \, d\nu(z) d\nu(w).$$

The quantity $c^0(K)$ is called the *homogeneous capacity* of K.

Lemma 3.12. *If v is archimedean and $K \subset \mathbb{C}^2$ is compact, then $d_\infty^0(K) = c^0(K)^{[k_v:\mathbb{R}]}$.*

Proof: Note that $[k_v : \mathbb{R}] = 1$ or 2, according as $k_v \cong \mathbb{R}$ or $k_v \cong \mathbb{C}$. The power $[k_v : \mathbb{R}]$ arises because of our normalization of absolute values: $|x|_v = |x|^{[k_v:\mathbb{R}]}$ for all $x \in \mathbb{C}$. If we replace $|x|_v$ with $|x|$ in the definition of $d_\infty^0(K)$, it suffices to show that $d_\infty^0(K) = c^0(K)$.

By a general fact about measures proved in Lemma 3.46 below, if ν_n is a probability measure supported equally on $z_1, \ldots, z_{N_n} \in K$, with $z_i \wedge z_j \neq 0$ for all $i \neq j$, and if $\nu_n \to \nu$ weakly on K, then

$$\liminf_{n\to\infty} \iint_{\mathbb{C}^2 \times \mathbb{C}^2 \backslash (\text{Diag})} -\log |z \wedge w|\, d\nu_n(z)d\nu_n(w) \;\geq\; \iint_{\mathbb{C}^2 \times \mathbb{C}^2} -\log |z \wedge w| d\nu(z)d\nu(w) \;. \tag{3.13}$$

Let $n \geq 2$, and define $D_n = -\log d_n^0(K)$. Given any $z_1, \ldots, z_n \in K$, we have

$$\frac{1}{n(n-1)} \sum_{i\neq j} -\log |z_i \wedge z_j| \;\geq\; D_n$$

by definition. Integrating this inequality against an arbitrary measure $\mu \in \mathbb{P}(K)$, we obtain

$$\frac{1}{n(n-1)} \sum_{i\neq j} \iint -\log |z_i \wedge z_j|\, d\mu(z_i)d\mu(z_j) \;\geq\; D_n$$

for all n, and therefore $c^0(K) \leq d_\infty^0(K)$.

For the other direction, choose $w_1, \ldots, w_n \in K$ such that

$$n(n-1)D_n \;=\; \sum_{i\neq j} -\log |w_i \wedge w_j| \;.$$

Without loss of generality, we may assume that $w_i \wedge w_j \neq 0$ for all $i \neq j$. Define the measure ν_n to be the discrete measure on K supported equally on each of the points w_i, i.e.,

$$\nu_n \;:=\; \frac{1}{n} \sum_i \delta_{w_i}.$$

By passing to a subsequence if necessary, we may assume that ν_n converges weakly to some probability measure ν on K. Noting that

$$\frac{n-1}{n} D_n \;=\; \frac{1}{n^2} \sum_{i\neq j} -\log |w_i \wedge w_j| \;=\; \iint_{\mathbb{C}^2 \times \mathbb{C}^2 \backslash (\text{Diag})} -\log |z \wedge w|\, d\nu_n(z)d\nu_n(w),$$

it follows from (3.13) and the definition of $c^0(K)$ that

$$\lim_{n\to\infty} D_n \;\geq\; I(\nu) \;\geq\; -\log c^0(K),$$

so that $d_\infty^0(K) \leq c^0(K)$ as desired. $\square$

We have introduced the homogeneous transfinite diameter by analogy with the relation between the classical transfinite diameter and logarithmic capacity over $\mathbb{C}$. The integral defining the homogeneous capacity is difficult to extend to nonarchimedean places. However, the transfinite diameter generalizes directly.

We will now give a formula for $d_\infty^0(K_{F,v})$ in terms of resultants, motivated by the following result ([DeM, Theorem 1.5]):

Theorem 3.14 (DeMarco)**.** *Suppose $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ for some homogeneous polynomials $F_1, F_2 \in \mathbb{C}[x, y]$ of degree d with no common linear factor, and let K_F be the filled Julia set of F in $\mathbb{C}^2$. Then*

$$c^0(K_F) \;=\; |\mathrm{Res}(F)|^{-1/d(d-1)}. \tag{3.15}$$

The proof given in [DeM], which involves both algebraic and analytic ingredients, and does not carry over easily to the nonarchimedean setting. One of our main results is the following adelic generalization of DeMarco's theorem:

Theorem 3.16. *Suppose $F = (F_1, F_2) : \mathbb{A}^2(\overline{k}) \to \mathbb{A}^2(\overline{k})$ for some homogeneous polynomials $F_1, F_2 \in k[x, y]$ of degree d with no common linear factor over $\overline{k}$. For $v \in M_k$, let $K_{F,v}$ be the filled Julia set of F in $\mathbb{C}_v^2$. Then*

$$d_\infty^0(K_{F,v}) \;=\; |\mathrm{Res}(F)|_v^{-1/d(d-1)}. \tag{3.17}$$

The proof, which is given in Section 6, requires the development of a considerable amount of capacity-theoretic machinery. It is completely independent of DeMarco's proof. The product formula yields the following corollary, a key ingredient in our proof of Theorem 2.3:

Corollary 3.18.

$$\sum_{v \in M_k} \log d_\infty^0(K_{F,v}) \;=\; 0 \ .$$

Remark 3.19. When $\varphi(z)$ is a polynomial of degree d with leading coefficient a_d, and $K_{\varphi,v} \subseteq \mathbb{C}_v$ is the v-adic filled Julia set of φ (as defined in [BH]), formula (3.17) specializes to the formula

$$c(K_{\varphi,v}) \;=\; |a_d|_v^{-1/(d-1)},$$

which was first proved in [BH].

3.4. **The Arakelov-Green's function and Local heights.** In this subsection we will construct a two-variable Green's function $g_{\varphi,v}(z, w)$ for the dynamical system associated to φ. It arises as a function on $\mathbb{C}_v^2$ which is invariant under scaling, and therefore descends to a function on $\mathbb{P}^1(\mathbb{C}_v)$. We will see that the descended function gives a continuously varying one-parameter family (indexed by $w \in \mathbb{P}^1(\mathbb{C}_v)$) of Call-Silverman local height functions.

For notational convenience, write $c_v(F) := |\mathrm{Res}(F)|_v^{-1/d(d-1)}$.

If $v \in M_k$ and $z, w \in \mathbb{C}_v^2$ are linearly independent over $\mathbb{C}_v$, define

$$G_{F,v}(z, w) \;:=\; -\log |z \wedge w|_v + \hat{H}_{F,v}(z) + \hat{H}_{F,v}(w) + \log c_v(F). \tag{3.20}$$

Recall that in the archimedean case, $\mathrm{SU}(2) = \{\theta \in SL(2, \mathbb{C}) : {}^t\overline{\theta}\theta = 1\}$ is the group preserving the both the norm $\|z\|_v$ on $\mathbb{C}^2$ and the alternating product $z \wedge w$, while in the nonarchimedean case if $\hat{\mathcal{O}}_v$ denotes the ring of integers of $\mathbb{C}_v$, then $SL(2, \hat{\mathcal{O}}_v)$ is the group preserving $\|z\|_v$ and $z \wedge w$ on $\mathbb{C}_v^2$. Write $\theta(F) = \theta \circ F \circ \theta^{-1}$.

We note the following properties of $G_{F,v}$.

Lemma 3.21.

A) $G_{F,v}$ *is doubly scale-invariant, in the sense that if $\alpha, \beta \in \mathbb{C}_v^*$, then*

$$G_{F,v}(\alpha z, \beta w) \;=\; G_{F,v}(z, w).$$

B) *For $\gamma \in \mathbb{C}_v^*$, we have*

$$G_{\gamma F,v}(z, w) \;=\; G_{F,v}(z, w).$$

C) *If v is archimedean, then for each $\theta \in \mathrm{SU}(2)$,*

$$G_{\theta(F),v}(\theta(z), \theta(w)) \;=\; G_{F,v}(z, w) \;.$$

If v is nonarchimedean, then for each $\theta \in SL(2, \hat{\mathcal{O}}_v)$

$$G_{\theta(F),v}(\theta(z), \theta(w)) \;=\; G_{F,v}(z, w) \;.$$

Proof: Part A) follows immediately from the fact that

$$\log |\alpha z \wedge \beta w|_v = \log |z \wedge w|_v + \log |\alpha|_v + \log |\beta|_v$$

and from the fact that $\hat{H}_{F,v}$ scales logarithmically.

Part B) follows from the following two easily verified facts:

(a) $\hat{H}_{\gamma F,v}(z) = \hat{H}_{F,v}(z) + \frac{1}{d-1} \log |\alpha|_v$.

(b) $|\mathrm{Res}(\gamma F)|_v = |\mathrm{Res}(F)|_v |\gamma|_v^{2d}$.

Part C) follows from the fact that the group $\mathrm{SU}(2)$ (resp. $SL(2, \hat{\mathcal{O}}_v)$) preserves $z \wedge w$, $\|z\|_v$, and $|\mathrm{Res}(F)|_v$. To see that $\mathrm{Res}(\theta(F)) = \mathrm{Res}(F)$, note first that manipulating the determinant defining $\mathrm{Res}(\theta \circ F)$ shows that $\mathrm{Res}(\theta \circ F) = \mathrm{Res}(F)$. On the other hand, if $F_1(z) = \prod_{i=1}^d (z \wedge a_i)$ and $F_2(z) = \prod_{j=1}^d (z \wedge b_j)$, then $\mathrm{Res}(F) = \pm \prod_{i,j} (a_i \wedge b_j)$. A simple computation shows that $\theta^{-1}(z) \wedge a_i = z \wedge \theta(a_i)$ and $\theta^{-1}(z) \wedge b_j = z \wedge \theta(b_j)$. Since $\theta(a_i) \wedge \theta(b_j) = a_i \wedge b_j$, it follows that $\mathrm{Res}(F \circ \theta^{-1}) = \mathrm{Res}(F)$. $\square$

In particular, $G_{F,v}$ descends to a well-defined function $g_{\varphi,v}(z, w)$ on $\mathbb{P}^1(\mathbb{C}_v)$: for any $z, w \in \mathbb{P}^1(\mathbb{C}_v)$ and any lifts $\tilde{z}, \tilde{w} \in \mathbb{C}_v^2$

$$g_{\varphi,v}(z, w) \;=\; -\log |\tilde{z} \wedge \tilde{w}|_v + \hat{H}_{F,v}(\tilde{z}) + \hat{H}_{F,v}(\tilde{w}) + \log c_v(F). \tag{3.22}$$

If $z \neq w$ then the right-hand side of (3.22) is finite; if $z = w$ then we define $g_{\varphi,v}(z, z) := +\infty$.

We will now establish another fact needed for the proof of Theorem 2.3. Define

$$\Gamma_{\varphi,v} \;=\; \liminf_{n \to \infty} \inf_{z_1,\dots,z_n \in \mathbb{P}^1(\mathbb{C}_v)} \frac{1}{n(n-1)} \sum_{i \neq j} g_{\varphi,v}(z_i, z_j). \tag{3.23}$$

Lemma 3.24 (Positivity). *For each $v \in M_k$, we have $\Gamma_{\varphi,v} \geq 0$.*

Proof: Let $\epsilon > 0$. Choose the lifts of the points z_i in the statement of the Lemma so that

$$-\epsilon \;<\; \hat{H}_{F,v}(\tilde{z}_i) \;\leq\; 0$$

for all i. This is possible because $\hat{H}_{F,v}$ scales logarithmically and the set $\{\log |\alpha|_v \; : \; \alpha \in \mathbb{C}_v^*\}$ is dense in $\mathbb{R}$.

In particular, $\tilde{z}_i \in K_{F,v}$ for all i. By the definition of the homogeneous transfinite diameter,

$$\liminf_{n \to \infty} \inf_{z_1,\dots,z_n \in \mathbb{P}^1(\mathbb{C}_v)} \frac{1}{n(n-1)} \sum_{i \neq j} -\log |\tilde{z}_i \wedge \tilde{z}_j|_v \;\geq\; -\log d_\infty^0(K_{F,v}) \;.$$

But $d_\infty^0(K_{F,v}) = c_v(F)$ by Theorem 3.16. Therefore we obtain the inequality

$$\liminf_{n \to \infty} \inf_{z_1,\dots,z_n \in \mathbb{P}^1(\mathbb{C}_v)} \frac{1}{n(n-1)} \sum_{i \neq j} g_{\varphi,v}(z_i, z_j) \;>\; -2\epsilon \;.$$

Since $\epsilon > 0$ was arbitrary, this gives the desired result. $\square$

Remark 3.25. Later, in Corollary 4.13, we will see via a global argument that $\Gamma_{\varphi,v} = 0$ for each v.

Next we will show that $g_{\varphi,v}(z,w)$ forms a one-parameter family of Call-Silverman local heights (see [CS],[CG]). Recall that a function $\hat{h}_{\varphi,v,D} : \mathbb{P}^1(\mathbb{C}_v) \setminus \mathrm{supp}(D) \to \mathbb{R}$ is called a Call-Silverman *canonical local height function* for φ, relative to the divisor D, if it is a Weil local height associated to D, and if there exists a rational function f on $\mathbb{P}^1$ over $\mathbb{C}_v$ with $\mathrm{div}(f) = \varphi^* D - d \cdot D$ such that

$$\hat{h}_{\varphi,v,D}(\varphi(z)) \;=\; d \cdot \hat{h}_{\varphi,v,D}(z) - \log |f|_v$$

for all $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus (\mathrm{supp}(D) \cup \mathrm{supp}(\varphi^*(D)))$. It is proved in [CS] that a canonical local height function exists for every divisor D, and is unique up to an additive constant.

Choose coordinates on $\mathbb{P}^1(\mathbb{C}_v)$ in such a way that ∞ corresponds to the point $[0:1]$ and 0 corresponds to $[1:0]$. Let $V_\infty = \mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$, so that every $z = (z_0 : z_1) \in V_\infty$ can be expressed uniquely as $[1 : T(z)]$ with $T(z) = z_1/z_0 \in \mathbb{C}_v$.

Define $\hat{h}_{F,v,(\infty)} : V_\infty \to \mathbb{R}$ by

$$\hat{h}_{F,v,(\infty)}(z) \;=\; \hat{H}_{F,v}(1, T(z)),$$

and note that for $z \in V_\infty$, we have $(1, T(z)) \wedge (0,1) = 1$, so

$$g_{\varphi,v}(z,w) = \begin{cases} -\log |T(z) - T(w)| + \hat{h}_{F,v,(\infty)}(z) + \hat{h}_{F,v,(\infty)}(w) + \log c_v(F) & w \neq \infty \\ \hat{h}_{F,v,(\infty)}(z) + \hat{H}_{F,v}((0,1)) + \log c_v(F) & w = \infty \end{cases} \tag{3.26}$$

For $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus (\{\infty\} \cup \varphi^{-1}(\infty))$, we have the identity

$$F^{(n-1)}(1, T(\varphi(z))) \;=\; F^{(n-1)}\left(1, \frac{F_2(1,T(z))}{F_1(1,T(z))}\right) \;=\; \frac{F^{(n)}(1,T(z))}{F_1(1,T(z))^{d^{n-1}}}. \tag{3.27}$$

Taking logarithms in (3.27) and letting $n \to \infty$ gives the functional equation

$$\hat{h}_{F,v,(\infty)}(\varphi(z)) \;=\; d \cdot \hat{h}_{F,v,(\infty)}(z) - \log |F_1(1,T(z))|_v, \tag{3.28}$$

which is valid for all $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus (\{\infty\} \cup \varphi^{-1}(\infty))$.

By Remark 3.6, there exists a constant $C > 0$ such that

$$|\hat{h}_{F,v,(\infty)}(z) - \log \max(1, |T(z)|_v)| \;\leq\; C \tag{3.29}$$

for all $z \in V_\infty$.

Equations (3.28) and (3.29) show that $\hat{h}_{F,v,(\infty)}$ is a Call-Silverman canonical local height function on $\mathbb{P}^1(\mathbb{C}_v)$ for φ relative to the divisor $D = (\infty)$. By (3.26), the function $g_{\varphi,v}(z,\infty)$ is also a Call-Silverman canonical local height for φ relative to (∞), since $g(z,\infty)$ and $\hat{h}_{F,v,(\infty)}(z)$ differ by a constant. More generally, using the fact that $\mathrm{SU}(2)$ (resp. $\mathrm{SL}(2,\hat{\mathcal{O}}_v)$) acts transitively on $\mathbb{P}^1(\mathbb{C}_v)$, it follows that for each $w \in \mathbb{P}^1(\mathbb{C}_v)$, the function $g_{\varphi,v}(z,w)$ is a Call-Silverman local height for φ relative to the divisor $D = (w)$. Thus $g_{\varphi,v}(z,w)$ is a continuously varying one-parameter family of local heights.

As a concrete example, take $w = 0$. In the affine patch $V_0 = \mathbb{P}^1(\mathbb{C}_v) \setminus \{0\}$, every point $z \in V_2$ can be represented uniquely as $[U(z) : 1]$ with $U(z) = z_0/z_1 \in \mathbb{C}_v$.

Define $\hat{h}_{F,v,(0)}(z) : V_0 \to \mathbb{R}$ by

$$\hat{h}_{F,v,(0)}(z) \;=\; \hat{H}_{F,v}(U(z), 1) \,,$$

so that for $z \in V_0$, we have

$$g_{\varphi,v}(z,w) = \begin{cases} -\log|U(z) - U(w)| + \hat{h}_{F,v,(0)}(z) + \hat{h}_{F,v,(0)}(w) + \log c_v(F) & w \neq 0 \\ \hat{h}_{F,v,(0)}(z) + \hat{H}_{F,v}((0,1)) + \log c_v(F) & w = 0. \end{cases} \tag{3.30}$$

Then for $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus (\{0\} \cup \varphi^{-1}(0))$, the identity

$$F^{(n-1)}(U(\varphi(z)), 1) \;=\; F^{(n-1)}\left(\frac{F_1(U(z),1)}{F_2(U(z),1)}, 1\right) \;=\; \frac{F^{(n)}(U(z),1)}{F_2(U(z),1)^{d^{n-1}}},$$

gives the functional equation

$$\hat{h}_{F,v,(0)}(\varphi(z)) \;=\; d \cdot \hat{h}_{F,v,(0)}(z) - \log|F_2(U(z),1)|_v, \tag{3.31}$$

valid for all $z \in \mathbb{P}^1(\mathbb{C}_v) \setminus (\{0\} \cup \varphi^{-1}(0))$. We also have

$$|\hat{h}_{F,v,(0)}(z) - \log\max(|U(z)|_v, 1)| \;\leq\; C$$

for all $z \in V_0$.

Finally, note that letting n tend to infinity in the identity

$$F^{(n)}(U(z), 1) \;=\; F^{(n)}(1, T(z))/T(z)^{d^n}$$

and taking logarithms gives

$$\hat{h}_{F,v,(0)}(z) \;=\; \hat{h}_{F,v,(\infty)}(z) - \log|T(z)|_v \tag{3.32}$$

for all $z \in V_0 \cap V_\infty = \mathbb{P}^1(\mathbb{C}_v) \setminus \{0, \infty\}$.

3.5. Arakelov Green's functions and the canonical measure.

In this subsection, we will show that $\frac{1}{\log(q_v)} g_{\varphi,v}(z,w)$ is in fact an Arakelov Green's function. This means showing that for each w, the Laplacian of $g_{\varphi,v}(z,w)$ satisfies

$$\frac{1}{\log(q_v)} \Delta(g_{\varphi,v}(z,w)) \;=\; \delta_w(z) - \mu_{\varphi,v}(z),$$

where $\mu_{\varphi,v}$ is a probability measure, independent of w. As will be explained below, in the nonarchimedean case the Laplacian is taken on the Berkovich space $\mathbb{P}^1_{\mathrm{Berk},v}$. In the archimedean case, $\mu_{\varphi,v}$ turns out to be the canonical measure supported on the Julia set of φ (see §2.1). Thus "the canonical measure is the minus Laplacian of the local height".

The measures $\mu_{\varphi,v}$ play a central role in our theory: they are the target measures in our main equidistribution theorem, Theorem 2.3.

In the archimedean case, for any Riemann surface $X/\mathbb{C}$, we define an Arakelov Green's function to be a function $g(z,w) : X(\mathbb{C}) \times X(\mathbb{C}) \to \mathbb{R} \cup \{\infty\}$ which satisfies the following two conditions:

(RS1) (Continuity) The function $g(z,w)$ is a continuous as a function from $X(\mathbb{C}) \times X(\mathbb{C})$ to the extended reals, and is finite off the diagonal.

(RS2) (Differential equation) There is a probability measure μ on $X(\mathbb{C})$ such that for each fixed w, $g(z,w)$ satisfies the distributional identity

$$\Delta_z g(z,w) \;=\; \delta_w(z) - \mu(z).$$

Conditions (RS1) and (RS2) imply that $g(z,w)$ is symmetric and bounded below, with a logarithmic singularity along the diagonal. These two conditions determine the function $g(z,w)$ up to an additive constant. There is a canonical way to normalize it: if

(RS3) (Normalization) $\iint g(z,w)\,d\mu(z)d\mu(w) = 0$,

we will say $g(z,w)$ is a *normalized* Arakelov Green's function. In any case, a non-normalized Arakelov Green's function still satisfies

$$\iint g(z,w)\,d\mu(z)d\mu(w) \; < \; \infty \; . \tag{3.33}$$

As noted in [CR] (see also Lemma 5.14 below), if $g(z,w)$ satisfies (RS1) and (RS2), then differentiating $\lambda(z) := \int g(z,w)\,d\mu(w)$ under the integral sign shows that $\lambda(z)$ is harmonic everywhere and therefore constant. Thus (RS3) is equivalent to the following apparently stronger condition:

(RS3)$'$ (Strong Normalization) $\int g(z,w)\,d\mu(w) \equiv 0$.

Remark 3.34. This definition of an Arakelov Green's function, taken from [R1] (see also [R2] and [Ma]), is slightly looser than the one commonly used in the literature, where the measure μ is required to be a smooth positive $(1,1)$-form ω with total mass 1.

The operator $\Delta = -dd^c$ on $X(\mathbb{C})$ is to be considered in the distributional sense. In local coordinates, if f is $\mathcal{C}^2$ then in terms of the standard real Laplacian we have

$$\Delta f = -\frac{1}{2\pi}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy \; .$$

The fact that the distributional Laplacian of $g(z,w)$ is a negative measure on $X(\mathbb{C})\backslash\{w\}$ means that the restriction of $g(z,w)$ to $X(\mathbb{C})\backslash\{w\}$ is subharmonic. It should also be noted that the continuity of $g(z,w)$ imposes conditions on the measure μ; in particular, μ cannot have any point masses. This follows from the Riesz Decomposition theorem ([Ts], Theorem II.24, p.45): on any local coordinate patch $V \subset X(\mathbb{C})\backslash\{w\}$, there is a harmonic function $h_V(z)$ such that for all $z \in V$

$$g(z,w) \;=\; h_V(z) + \int_V \log|z-x|\,d\mu(x)$$

If μ had a point mass at some $p \in V$, then we would have $g(p,w) = -\infty$, a contradiction.

The continuity of $g(z,w)$ also shows that $u_V(z) = \int_V \log|z-x|\,d\mu(x)$ is a continuous function of z on V. Therefore μ must be log-continuous, in the following sense:

Definition 3.35. A bounded Borel measure μ is *log-continuous* if for each $p \in X(\mathbb{C})$ there is a neighborhood V of p such that

$$u_V(z) \; := \; \int_V \log(|z-x|)\,d\mu(x)$$

is continuous on V.

In [R1] this concept was called *log-finiteness*. However, the terminology log-continuous used here seems more appropriate.

In the nonarchimedean case, there is also a notion of an Arakelov Green's function. In theory, one could define Arakelov Green's functions on an arbitrary Berkovich curve over $\mathbb{C}_v$, but we restrict ourselves here to the case $X = \mathbb{P}^1_{\mathrm{Berk},v}$. We only sketch the basic framework; for further details, see [RB, §4 –§7].

There is a class of functions on $\mathbb{P}^1_{\mathrm{Berk},v}$, called *functions of bounded differential variation*, for which it is possible to define a measure-valued Laplacian; this class is denoted $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk},v})$ (see [RB], §5.3). The Laplacian is defined first for functions on finitely branched subgraphs of

$\mathbb{P}^1_{\mathrm{Berk},v} \backslash \mathbb{P}^1(\mathbb{C}_v)$ via the construction in ([BR], §4) which generalizes the approaches of [CR] and [Zh1]. It is then extended by a limiting process to functions on open subdomains of $\mathbb{P}^1_{\mathrm{Berk},v}$, using the Riesz Representation theorem. There are analogues of harmonic functions and subharmonic functions on $\mathbb{P}^1_{\mathrm{Berk},v}$. Harmonic functions satisfy a maximum principle ([RB], Proposition 5.14), a Poisson formula ([RB], Proposition 5.18), and Harnack's principle ([RB], Proposition 5.24). Subharmonic functions are functions which locally belong to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk},v})$ and have non-negative Laplacian ([RB], Proposition 6.1). They have stability properties similar to classical subharmonic functions ([RB], Proposition 6.11) and satisfy a maximum principle ([RB], Proposition 6.15), a comparison theorem ([RB], Proposition 6.16), and a Riesz Decomposition Theorem ([RB], Proposition 6.19). The pullback of a subharmonic function by a rational map is subharmonic ([RB], Proposition 7.13). In brief, [RB] provides all the tools necessary carry through arguments of classical potential theory on $\mathbb{P}^1_{\mathrm{Berk},v}$.

For a rational function φ acting on $\mathbb{P}^1_{\mathrm{Berk},v}$, there is a theory of multiplicities at points of $\mathbb{P}^1_{\mathrm{Berk},v}$, extending the usual algebraic multiplicities on $\mathbb{P}^1(\mathbb{C}_v)$ (see [RB], Proposition 7.2). Given a Borel measure μ on $\mathbb{P}^1_{\mathrm{Berk},v}$, this makes it possible to define pushforward and pullback measures $\varphi_* \mu$ and $\varphi^* \mu$ with the usual formal properties ([RB], §7.3).

A (Berkovich) Arakelov Green's function is a function $g(z, w) : \mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v} \to \mathbb{R} \cup \{\infty\}$ such that

(B1) (Semicontinuity) The function $g(z, w)$ is finite and continuous off the diagonal, and is strongly lower-semicontinuous on the diagonal, in the sense that for each $z \in \mathbb{P}^1_{\mathrm{Berk},v}$

$$g(z, z) \;=\; \liminf_{\substack{(x,y) \to (z,z) \\ x \neq y}} g(x, y) \;.$$

(B2) (Differential equation) For each $w \in \mathbb{P}^1_{\mathrm{Berk}}$, $g(z, w)$ belongs to $\mathrm{BDV}(\mathbb{P}^1_{\mathrm{Berk},v})$. Furthermore, there is a probability measure μ on $\mathbb{P}^1_{\mathrm{Berk},v}$ such that for each w, $g(z, w)$ satisfies the identity

$$\Delta_z g(z, w) \;=\; \delta_w(z) - \mu(z).$$

As in the archimedean case, conditions (B1) and (B2) imply that $g(z, w)$ is symmetric and bounded below. The semicontinuity along the diagonal is a technical condition which arises naturally from properties of the space $\mathbb{P}^1_{\mathrm{Berk},v}$ (see [RB], Proposition 3.1). Together, (B1) and (B2) determine $g(z, w)$ up to an additive constant by the maximum principle ([RB], Proposition 5.14). If in addition

(B3) (Normalization) $\iint g(z, w) \, d\mu(z) d\mu(w) \;=\; 0$,

we will say $g(z, w)$ is a *normalized* Berkovich Arakelov Green's function.

Again, our assumption that $g(z, w)$ is continuous off the diagonal means that μ is log-continuous (the precise definition, and proof, are given in [RB], Proposition 7.15). And as in the archimedean case, log-continuity implies that μ has no point masses on $\mathbb{P}^1(\mathbb{C}_v)$. However, it can have point masses on $\mathbb{P}^1_{\mathrm{Berk},v} \backslash \mathbb{P}^1(\mathbb{C}_v)$ (see Example 3.43 below).

The function $g_{\varphi,v}(z, w)$ has a natural extension 'by continuity' to $\mathbb{P}^1_{\mathrm{Berk},v}$; for details, see ([RB], §7.5). We will write $g_{\varphi,v}(z, w)$ for both the function on $\mathbb{P}^1(\mathbb{C}_v)$ constructed above, and its extension to $\mathbb{P}^1_{\mathrm{Berk},v}$.

Recall that q_v is the order of the residue field of k_v. We will now show that for each v, the function $\frac{1}{\log(q_v)} g_{\varphi,v}(z, w)$ is an Arakelov Green's function. The probability measure $\mu_{\varphi,v}$

associated to $g_{\varphi,v}(z,w)$ (i.e., the measure occurring in (RS2) or (B2)) plays a key role in our theory. In the nonarchimedean case, $\mu_{\varphi,v}$ is a measure on $\mathbb{P}^1_{\mathrm{Berk},v}$; in the archimedean case, it is a measure on $\mathbb{P}^1(\mathbb{C})$. As noted earlier, $\mathbb{P}^1_{\mathrm{Berk}}/\mathbb{C} \cong \mathbb{P}^1(\mathbb{C})$, so in fact we can view $\mu_{\varphi,v}$ as a measure on $\mathbb{P}^1_{\mathrm{Berk},v}$ for all v.

Proposition 3.36.
  A) *For each $v \in M_k$, the function $\frac{1}{\log(q_v)}g_{\varphi,v}(z,w)$ is an Arakelov Green's function associated to a log-continuous probability measure $\mu_{\varphi,v}$ on $\mathbb{P}^1_{\mathrm{Berk},v}$. For each $w \in \mathbb{P}^1(\mathbb{C}_v)$, the measure $\mu_{\varphi,v}$ is given locally on $V_w := \mathbb{P}^1_{\mathrm{Berk},v}\backslash\{w\}$ by*

$$\mu_{\varphi,v}|_{V_w} \;=\; -\frac{1}{\log(q_v)}\Delta g_{\varphi,v}(z,w). \qquad (3.37)$$

Furthermore, $\varphi^*\mu_{\varphi,v} = d \cdot \mu_{\varphi,v}$ and $\varphi_*\mu_{\varphi,v} = \mu_{\varphi,v}$.
  B) *If $v \in M_k$ is archimedean, then $\mu_{\varphi,v}$ coincides with the canonical measure on $\mathbb{P}^1(\mathbb{C})$ associated to φ by Lyubich and Freire-Lopes-Mañé.*

Proof: For v nonarchimedean, this is [RB, Theorem 7.14]. Henceforth assume v is archimedean. For part A), note first that both $\hat{h}_{F,v,(0)}(z)$ and $\hat{h}_{F,v,(\infty)}(z)$ are uniform limits of subharmonic functions, and are therefore subharmonic. Thus both $-\Delta\hat{h}_{F,v,(0)}(z)$ and $-\Delta\hat{h}_{F,v,(\infty)}(z)$ are nonnegative measures. Using the relation (3.32), we see that $\hat{h}_{F,v,(0)}(z)$ and $\hat{h}_{F,v,(\infty)}(z)$ differ by a harmonic function on $V_0 \cap V_\infty$, and therefore $\Delta\hat{h}_{F,v,(\infty)}(z) = \Delta\hat{h}_{F,v,(0)}(z)$ on $V_0 \cap V_\infty$. It follows that there is a non-negative measure $\mu_{\varphi,v}$ given locally by (3.37). By (3.26) and (3.30), for any w we have $\frac{1}{\log(q_v)}\Delta_z g_{\varphi,v}(z,w) = \delta_w(z) - \mu_{\varphi,v}$ on both V_0 and V_∞, and hence on $V_0 \cup V_\infty = \mathbb{P}^1(\mathbb{C})$, as desired.

The fact that $\mu_{\varphi,v}$ is a *probability* measure (i.e., that $\mu_{\varphi,v}(\mathbb{P}^1(\mathbb{C})) = 1$) follows immediately from the identity $\frac{1}{\log(q_v)}\Delta_z g_{\varphi,v}(z,w) = \delta_w(z) - \mu_{\varphi,v}$, since the distributional Laplacian of a function on $\mathbb{P}^1(\mathbb{C})$ always has total mass zero.

To see that $\varphi^*(\mu_{\varphi,v}) = d \cdot \mu_{\varphi,v}$, combine (3.28) and (3.31), using the fact that F_1 and F_2 have no common zeros in $\mathbb{C}^2$ by assumption. Finally, the relation $\varphi_*(\mu_{\varphi,v}) = \mu_{\varphi,v}$ follows formally from $\varphi^*(\mu_{\varphi,v}) = d \cdot \mu_{\varphi,v}$ using the fact that $\varphi_*(\varphi^*(\mu)) = d \cdot \mu$ for all measures μ on $\mathbb{P}^1(\mathbb{C})$.

For part B), recall from Theorem 2.1 that the canonical measure is the unique probability measure μ on $\mathbb{P}^1(\mathbb{C})$ with no point masses such that $\varphi^*(\mu) = d \cdot \mu$. As noted above, the continuity of $g_{\varphi,v}(z,w)$ off the diagonal implies that $\mu_{\varphi,v}$ has no point masses, and the functional equation $\varphi^*(\mu_{\varphi,v}) = d \cdot \mu_{\varphi,v}$ has been established in A). Hence $\mu_{\varphi,v}$ coincides with the canonical measure. $\qquad\square$

Remark 3.38. We will see in Corollary 4.13, as a consequence of global considerations, that $\frac{1}{\log(q_v)}g_{\varphi,v}(z,w)$ is in fact a *normalized* Arakelov Green's function for each v.

Assuming Remark 3.38, we can establish the following invariance property of $g_{\varphi,v}(z,w)$:

Corollary 3.39. *Given $w \in \mathbb{P}^1_{\mathrm{Berk},v}$, write $\varphi^*((w)) = \sum_{i=1}^r m_i(w_i)$. Then for all $z \in \mathbb{P}^1_{\mathrm{Berk},v}$,*

$$g_{\varphi,v}(\varphi(z),w) \;=\; \sum_{i=1}^r m_i\, g_{\varphi,v}(z,w_i) \ .$$

Proof: We will only give the proof in the archimedean case; in the nonarchimedean case the proof is formally identical, using properties of the Berkovich Laplacian.

Since $SU(2)$ acts transitively on $\mathbb{P}^1(\mathbb{C})$, we can assume without loss that $w = \infty$. By formula (3.26), there is a constant C_1 such that

$$g_{\varphi,v}(\varphi(z), \infty) = \hat{h}_{F,v,(\infty)}(\varphi(z)) + C_1 . \tag{3.40}$$

By the functional equation (3.28) of the Call-Silverman local height $\hat{h}_{F,v,(\infty)}$,

$$\hat{h}_{F,v,(\infty)}(\varphi(z)) = d \cdot \hat{h}_{F,v,(\infty)}(z) - \log |F_1(1, T(z))|_v . \tag{3.41}$$

Here $f(z) := F(1, T(z))$ is a polynomial with divisor $\mathrm{div}(f) = \sum m_i(w_i) - d \cdot (\infty)$, where $\sum m_i(w_i) = \varphi^*(\infty)$. We claim there is a constant C_2 such that

$$-\log(|f(z)|_v) = \sum m_i g_{\varphi,v}(z, w_i) - d \cdot g_{\varphi,v}(z, \infty) + C_2. \tag{3.42}$$

To see this, note that both sides have Laplacian equal to $\log(q_v)$ times $\sum m_i \delta_{w_i}(z) - d \cdot \delta_\infty(z)$; hence their difference is a function which is harmonic everywhere, thus constant. (In the nonarchimedean case this argument is justified by [RB], Proposition 5.14 and Lemma 5.12). Combining (3.40), (3.41) and (3.42) shows that

$$g_{\varphi,v}(\varphi(z), \infty) = \sum_{i=1}^{r} m_i g_{\varphi,v}(z, w_i) + C$$

for some constant C. Integrating the left-hand side against $\mu_{\varphi,v}$ and using the invariance property of $\mu_{\varphi,v}$, the fact that $\frac{1}{\log(q_v)} g_{\varphi,v}(z, w)$ is normalized, and (RS3)′, we have

$$\int g_{\varphi,v}(\varphi(z), \infty) \, d\mu_{\varphi,v}(z) = \int g_{\varphi,v}(z, \infty) \, d(\varphi_* \mu_{\varphi,v})(z) = \int g_{\varphi,v}(z, \infty) \, d\mu_{\varphi,v}(z) = 0 .$$

Computing the integral of the right-hand side, we get C. Therefore $C = 0$ as desired. □

Example 3.43. Recall from [CS] that φ is said to have *good reduction* at a place v if it can be written as $\varphi(T) = G_2(T)/G_1(T)$ where $G_1, G_2 \in \mathcal{O}_v(T)$ are such that the reduced polynomials $g_1 = G_1 \mod (m_v)$, $g_2 = G_2 \mod (m_v)$ are nonzero and coprime, with $\max(\deg(g_1), \deg(g_2)) = d$.

If φ has good reduction at v, then by Example 7.2 of [RB], $\mu_{\varphi,v}$ is the discrete measure supported at the *Gauss point* ζ_0 of $\mathbb{P}^1_{\mathrm{Berk},v}$, and

$$g_{\varphi,v}(z, w) = \begin{cases} -\log|z - w|_v + \log^+ |z|_v + \log^+ |w|_v & z, w \neq \infty \\ \log^+ |z|_v & w = \infty \\ \log^+ |w|_v & z = \infty. \end{cases} \tag{3.44}$$

3.6. **The Energy Minimization Principle.** If $g(z, w)$ is an Arakelov Green's function on $\mathbb{P}^1_{\mathrm{Berk},v}$ associated to the measure μ, we will often write $g_\mu(z, w)$ instead of $g(z, w)$. With this notation, $\frac{1}{\log(q_v)} g_{\varphi,v}(z, w) = g_{\mu_{\varphi,v}}(z, w)$.

Arakelov Green's functions on $\mathbb{P}^1_{\mathrm{Berk},v}$ satisfy the following important energy minimization principle:

Theorem 3.45. *Let v be a place of k, and let $g_\mu(z,w)$ be an Arakelov Green's function on $\mathbb{P}^1_{\mathrm{Berk},v}$ whose associated measure μ is log-continuous. Define the "energy functional" $I_\mu(\nu)$ on the space $\mathbb{P}$ of probability measures on $\mathbb{P}^1_{\mathrm{Berk},v}$ by the formula*

$$I_\mu(\nu) \ := \ \iint_{\mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v}} g_\mu(z,w) \, d\nu(z)d\nu(w).$$

Then $I_\mu(\nu) \geq I_\mu(\mu)$ for all $\nu \in \mathbb{P}$, with equality if and only if $\nu = \mu$.

In the archimedean case, Theorem 3.45 will be proved in §5 as a consequence of the more general Theorem 5.8; in the nonarchimedean case, it is proved in [RB, Theorem 7.20].

3.7. **Discrete approximations to the energy integral.** In this section, v denotes an arbitrary place of k, and we work on the Berkovich space $\mathbb{P}^1_{\mathrm{Berk},v}$. Recall that if v is archimedean, the space $\mathbb{P}^1_{\mathrm{Berk},v}$ is just $\mathbb{P}^1(\mathbb{C})$.

The following lemma enables us to apply the energy minimization principle in a useful way to discrete measures. We state it abstractly because it was also used in Lemma 3.12.

Lemma 3.46. *Let (X,ν) be a measure space, with ν a probability measure. Let $\{S_n\}_{n\geq 1}$ be a sequence of finite subsets of X, and for each n let δ_n be the discrete probability measure supported equally at all elements of S_n. Suppose the measures δ_n converge weakly to ν. Let $g : X \times X \to \mathbb{R} \cup \{\infty\}$ be a function which is finite, continuous, and bounded from below on $X \times X \backslash (\mathrm{Diag})$. Then*

$$\liminf_{n\to\infty} \iint_{X \times X \backslash (\mathrm{Diag})} g(z,w) \, d\delta_n(z)d\delta_n(w) \ \geq \ \iint_{X \times X} g(z,w) \, d\nu(z)d\nu(w).$$

Proof:

Define $N_n := \#S_n$. For any fixed real number $M > 0$, we have

$$\iint_{(\mathrm{Diag})} \min\{M, g(x,y)\} \, d\delta_n(x)d\delta_n(y) \ = \ \frac{1}{N_n} \cdot M \tag{3.47}$$

by the definition of δ_n, and therefore

$$\liminf_{n\to\infty} \iint_{X \times X \backslash (\mathrm{Diag})} g(x,y) \, d\delta_n(x)d\delta_n(y)$$

$$\geq \lim_{M\to\infty} \liminf_{n\to\infty} \iint_{X \times X \backslash (\mathrm{Diag})} \min\{M, g(x,y)\} \, d\delta_n(x)d\delta_n(y) \qquad (\text{since } (*) \geq \min\{M, (*)\})$$

$$= \lim_{M\to\infty} \liminf_{n\to\infty} \iint_{X \times X} \min\{M, g(x,y)\} \, d\delta_n(x)d\delta_n(y) \qquad (\text{by } (3.47))$$

$$= \lim_{M\to\infty} \iint_{X \times X} \min\{M, g(x,y)\} \, d\nu(x)d\nu(y) \qquad (\delta_n \to \nu \text{ weakly})$$

$$= \iint_{X \times X} g(x,y) \, d\nu(x)d\nu(y) \qquad (\text{monotone convergence theorem}).$$

$$\square$$

Let $g_\mu(z,w)$ be an Arakelov Green's function on $\mathbb{P}^1_{\mathrm{Berk},v}$ with associated log-continuous measure μ. We now introduce a quantity $D_\infty(\mu)$ analogous to the (negative logarithm of the) classical transfinite diameter. For $n \geq 2$, define

$$D_n(\mu) \ := \ \inf_{z_1,\dots,z_n \in \mathbb{P}^1(\mathbb{C}_v)} \frac{1}{n(n-1)} \sum_{i \neq j} g_\mu(z_i, z_j).$$

Since $g_\mu(z, w)$ is bounded below and is finite off the diagonal, each $D_n(\mu)$ is a well-defined real number.

The proof of the following lemma is similar to that of Lemma 3.10.

Lemma 3.48. *The sequence $D_n(\mu)$ is non-decreasing.*

Proof: Take $n \geq 2$, fix $\varepsilon > 0$, and choose $w_1, \dots, w_{n+1}$ such that

$$\sum_{i \neq j} g_\mu(w_i, w_j) \ \leq \ n(n+1)(D_{n+1}(\mu) + \varepsilon) \ .$$

By the definition of $D_n(\mu)$, we have (for each $1 \leq m \leq n + 1$)

$$n(n-1)D_n(\mu) \ \leq \ \sum_{\substack{i,j \neq m \\ i \neq j}} g_\mu(w_i, w_j).$$

Adding together these $n + 1$ inequalities gives

$$(n+1)n(n-1)D_n(\mu) \ \leq \ (n-1)\sum_{i \neq j} g(w_i, w_j) \ \leq \ (n-1)n(n+1)(D_{n+1}(\mu) + \varepsilon) \ .$$

Since $\varepsilon > 0$ is arbitrary, $D_n(\mu) \leq D_{n+1}(\mu)$ as desired. $\square$

Define

$$D_\infty(\mu) \ = \ \lim_{n \to \infty} D_n(\mu) \ .$$

The following result is analogous to the equality of the transfinite diameter and the capacity of a compact set in classical complex potential theory.

Theorem 3.49. $D_\infty(\mu) = I_\mu(\mu)$.

Proof: Let $n \geq 2$. We first claim that for all $z_1, \dots, z_n \in \mathbb{P}^1_{\mathrm{Berk},v}$,

$$\frac{1}{n(n-1)} \sum_{i \neq j} g_\mu(z_i, z_j) \ \geq \ D_n(\mu) \ . \tag{3.50}$$

In the archimedean case this is immediate, since $\mathbb{P}^1_{\mathrm{Berk},v} = \mathbb{P}^1(\mathbb{C})$. To see it in the nonarchimedean case, first suppose the z_i are distinct, and note that by the continuity of $g_\mu(z, w)$ off the diagonal and the fact that $\mathbb{P}^1(\mathbb{C}_v)$ is dense in $\mathbb{P}^1_{\mathrm{Berk},v}$, for any $\varepsilon > 0$ there are points $x_1, \dots, x_n \in \mathbb{P}^1(\mathbb{C}_v)$ with

$$|g_\mu(z_i, z_j) - g_\mu(x_i, x_j)| \ < \ \varepsilon \ .$$

By definition we have $\frac{1}{n(n-1)} \sum_{i \neq j} g_\mu(x_i, x_j) \geq D_n(\mu)$, so letting $\varepsilon \to 0$ gives (3.50). The general case follows by the strong lower semicontinuity of $g_\mu(z, w)$ (see axiom (B1) for Berkovich Arakelov Green's functions).

Integrating (3.50) against $d\mu(z_1) \cdots d\mu(z_n)$, we see that

$$\frac{1}{n(n-1)} \sum_{i \neq j} \iint g_\mu(z_i, z_j) d\mu(z_i) d\mu(z_j) \ \geq \ D_n(\mu),$$

for all n, and therefore $I_\mu(\mu) \geq D_\infty(\mu)$.

For the other direction, for each n choose $w_1, \dots, w_n \in \mathbb{P}^1(\mathbb{C}_v)$ such that

$$\frac{1}{n(n-1)} \sum_{i \neq j} g_\mu(w_i, w_j) \ \leq \ D_n(\mu) + \frac{1}{n}$$

and let ν_n be the discrete measure supported equally on each of the points w_i, i.e.,

$$\nu_n := \frac{1}{n} \sum_i \delta_{w_i}.$$

By passing to a subsequence if necessary, we may assume that the ν_n converge weakly to some measure ν on $\mathbb{P}^1_{\text{Berk},v}$. Noting that

$$\frac{n-1}{n}\left(D_n(\mu) + \frac{1}{n}\right) \geq \frac{1}{n^2} \sum_{i \neq j} g_\mu(w_i, w_j) = \iint_{\mathbb{P}^1_{\text{Berk},v} \times \mathbb{P}^1_{\text{Berk},v} \backslash (\text{Diag})} g_\mu(w, z) \, d\nu_n d\nu_n \,,$$

it follows from Theorem 3.45 and Lemma 3.46 that

$$D_\infty(\mu) = \liminf_{n \to \infty} D_n(\mu) \geq I_\mu(\nu) \geq I_\mu(\mu)$$

as desired. $\square$

Remark 3.51. By the exact same arguments, one sees that Lemma 3.48 and Theorem 3.49 remain true in the archimedean case for an arbitrary Riemann surface, using Theorem 5.8 instead of Theorem 3.45.

4. Proof of the main equidistribution theorem

We now turn to the proof of Theorem 2.3. As will be seen, the theorem follows rather formally from the machinery developed above. Before giving the argument, we deal with some technical preliminaries.

4.1. Base change lemmas.

In this subsection, we formulate a lemma which relates local Arakelov Green's functions over different base fields.

Let φ be a rational function of degree $d \geq 2$ defined over the number field k. For $v \in M_k$, define $g_{\varphi,v}(z, w) := G_{F,v}(z, w)$ for some lift F of φ to $k[x, y] \times k[x, y]$, i.e., given $z, w \in \mathbb{P}^1(\overline{k})$, take lifts $\tilde{z}, \tilde{w}$ of z and w to $\overline{k}^2$; then

$$g_{\varphi,v}(z, w) = -\log|\tilde{z} \wedge \tilde{w}|_v + \hat{H}_{F,v}(\tilde{z}) + \hat{H}_{F,v}(\tilde{w}) + \log c_v(F),$$

where $c_v(F) = |\text{Res}(F)|^{-\frac{1}{d(d-1)}}$ as before. If k'/k is a finite extension, we can in a similar way define $g_{\varphi,v'}(z, w)$ for $v' \in M_{k'}$. We have:

Lemma 4.1. A) *The expression*

$$g_\varphi(z, w) := \frac{1}{[k' : \mathbb{Q}]} \sum_{v' \in M_{k'}} g_{\varphi,v'}(z, w)$$

is independent of the choice of a number field k' containing z and w, and therefore gives a well-defined function on $\overline{k} \times \overline{k} \backslash (\text{Diag})$.
   B) *For all $z, w \in \overline{k}$, $z \neq w$,*

$$g_\varphi(z, w) = \hat{h}_\varphi(z) + \hat{h}_\varphi(w).$$

C) *Let k' be a finite extension of k. Take $v \in M_k$, and let v' be a place of k' with $v' \mid v$. If S is a finite $\mathrm{Gal}(k'/k)$-invariant subset of k', then for all $z, w \in S$, $z \neq w$, the expression $\sum_{z \neq w \in S} g_{v'}(z, w)$ is independent of the place v', and*

$$\frac{1}{[k : \mathbb{Q}]} \left(\sum_{z \neq w \in S} g_{\varphi, v}(z, w) \right) = \frac{1}{[k' : \mathbb{Q}]} \sum_{v' \mid v} \left(\sum_{z \neq w \in S} g_{\varphi, v'}(z, w) \right).$$

D) *Let $z_1, \dots, z_N$ be the Galois conjugates of an element $z \in \mathbb{P}^1(\overline{\mathbb{Q}}) \backslash \mathbb{P}^1(\mathbb{Q})$. Then*

$$\frac{1}{[k : \mathbb{Q}]} \sum_{v \in M_k} \left(\frac{1}{N(N-1)} \sum_{i \neq j} g_{\varphi, v}(z_i, z_j) \right) = 2 \hat{h}_\varphi(z). \tag{4.2}$$

Proof: The proofs of A) and C) are straightforward consequences of our choice of normalizations for absolute values. B) follows from A) by the product formula (applied twice): if k'/k is a finite extension such that the lifts $\tilde{z}$ and $\tilde{w}$ are rational over k', then

$$\begin{aligned}
g_\varphi(z, w) &= \frac{1}{[k' : \mathbb{Q}]} \sum_{v' \in M_{k'}} g_{\varphi, v'}(z, w) \\
&= \frac{1}{[k' : \mathbb{Q}]} \sum_{v' \in M_{k'}} \left(-\log |\tilde{z} \wedge \tilde{w}|_{v'} + \hat{H}_{F, v'}(\tilde{z}) + \hat{H}_{F, v'}(\tilde{w}) + \log c_{v'}(F) \right) \\
&= \hat{h}_\varphi(z) + \hat{h}_\varphi(w)
\end{aligned}$$

since $\tilde{z} \wedge \tilde{w} = \tilde{z}_0 \tilde{w}_1 - \tilde{z}_1 \tilde{w}_0 \in (k')^*$ and $c_{v'}(F) = |\mathrm{Res}(F)|_{v'}^{-1/d(d-1)}$. Finally, D) follows from C) by summing both sides over all places v of k. $\square$

4.2. **Lemmas on double sums.** Before turning to Theorem 2.3 and its proof, we need two lemmas on doubly-indexed sums. The first is a discrete analogue of Fatou's lemma from real analysis:

Lemma 4.3. *Suppose $a_n^{(j)}$ is a doubly-indexed sequence of real numbers which satisfy the following two properties:*

(F1) *For each n, $\sum_j a_n^{(j)}$ converges.*

(F2) *There is a collection $\{M_j\}$ of real numbers, almost all zero, such that $a_n^{(j)} \geq -M_j$ for all j, n.*

Then

$$\sum_j \liminf_{n \to \infty} a_n^{(j)} \leq \liminf_{n \to \infty} \sum_j a_n^{(j)} \tag{4.4}$$

as extended real numbers.

Proof: Replacing $a_n^{(j)}$ by $a_n^{(j)} + M_j$ if necessary, we may assume without loss of generality that $a_n^{(j)} \geq 0$ for all n, j. The result now follows immediately from the usual version of Fatou's lemma (see [Roy], Theorem 4.3.9) applied to the sequence f_n of locally constant functions defined by $f_n(x) = a_n^{(j)}$ if $x \in [j, j+1)$. $\square$

The next lemma is a simple application of Lemma 4.3.

Lemma 4.5. *Suppose $a_n^{(j)}$ is a doubly-indexed sequence of real numbers which satisfy properties (F1) and (F2). Consider the following conditions, where $L, L_j \in \mathbb{R}$.*

$(L1)$ $\displaystyle\limsup_{n\to\infty}\sum_j a_n^{(j)} \leq L$,

$(L2)$ $\displaystyle\sum_j \liminf_{n\to\infty} a_n^{(j)} \geq L$,

$(L2)'$ $\displaystyle\liminf_{n\to\infty} a_n^{(j)} \geq L_j$.

Then:

A) *If $(L1)$ and $(L2)$ hold for some L, then $\lim_{n\to\infty} a_n^{(j)}$ exists for all j.*

B) *If $(L1)$ holds, and there are numbers L_j with with $\sum_j L_j \geq L$ such that $(L2)'$ holds for all j, then $\lim_{n\to\infty} a_n^{(j)} = L_j$ for all j.*

Proof: For any sequences a_n, b_n of real numbers which are bounded from below, it is easy to see that $\limsup(a_n + b_n) \geq \limsup(a_n) + \liminf(b_n)$.

For any index i, one therefore sees from Lemma 4.3 that

$$
\begin{aligned}
L &\geq \limsup\sum_j a_n^{(j)} \geq \limsup a_n^{(i)} + \liminf\sum_{j\neq i} a_n^{(j)} \\
&\geq \limsup a_n^{(i)} + \sum_{j\neq i}\liminf a_n^{(j)} \geq \limsup a_n^{(i)} - \liminf a_n^{(i)} + L ,
\end{aligned}
$$

which implies that $\limsup a_n^{(i)} \leq \liminf a_n^{(i)}$. Therefore $\lim_{n\to\infty} a_n^{(i)}$ exists for all i, which proves A).

For B), note that

$$
\begin{aligned}
L &\geq \limsup\sum_j a_n^{(j)} \geq \liminf\sum_j a_n^{(j)} \\
&\geq \sum_j\liminf a_n^{(j)} \geq \sum L_j \geq L.
\end{aligned}
$$

Therefore equality holds everywhere, so $\liminf a_n^{(j)} = L_j$ for all j. By A), we conclude that $\lim a_n^{(j)} = L_j$ for all j, which establishes B). $\qquad\square$

In applying Lemma 4.5, we will use the following easily verified properties of the collection of functions $g_{\varphi,v}(z,w)$:

(G1) For fixed $z, w \in \mathbb{P}^1(\overline{k})$ with $z \neq w$, we have $g_{\varphi,v}(z,w) = 0$ for almost all v.

(G2) For almost all v, we have $g_{\varphi,v}(z,w) \geq 0$ for all $z, w \in \mathbb{P}^1(\mathbb{C}_v)$.

4.3. **Pseudo-equidistribution.**

Let v be a place of k. If S is a finite subset of $\mathbb{P}^1(\mathbb{C}_v)$ of cardinality N, we define a discrete probability measure $\delta(S)$ on $\mathbb{P}^1(\mathbb{C}_v)$ by

$$
\delta(S) \; := \; \frac{1}{N}\sum_{z\in S}\delta_z \; .
$$

Note that if S is a subset of $\mathbb{P}^1(\overline{k})$, we can consider S as a subset of $\mathbb{P}^1(\mathbb{C}_v)$ for each $v \in M_k$, since we have fixed an embedding $\overline{k}\hookrightarrow\mathbb{C}_v$ for each v. If S is $\mathrm{Gal}(\overline{k}/k)$-stable, the resulting subset of $\mathbb{P}^1(\mathbb{C}_v)$ is independent of the choice of embedding.

Definition 4.6. A sequence of finite subsets $\{S_n\}_{n\geq 1}$ of $\mathbb{P}^1(\mathbb{C}_v)$ is *pseudo-equidistributed* with respect to $g_{\varphi,v}$ if $N_n = \#(S_n) \to \infty$ and

$$\lim_{n\to\infty} \frac{1}{N_n(N_n-1)} \sum_{\substack{z,w\in S_n \\ z\neq w}} g_{\varphi,v}(z,w) \;=\; 0 \; . \tag{4.7}$$

Recall that by Lemma 3.24 the minimal possible value for the left-hand side of (4.7) is 0. Thus, the sequence $\{S_n\}_{n\geq 1}$ is pseudo-equidistributed if and only if it achieves this minimum value.

Remark 4.8. This definition is a bit different from the definition of pseudo-equidistribution in ([BH]). It anticipates the fact, shown in Corollary 4.13 below, that $\frac{1}{\log(q_v)} g_{\varphi,v}(z,w)$ is a normalized Arakelov Green's function.

We now prove the following adelic pseudo-equidistribution result:

Theorem 4.9. *Let z_n be a sequence of distinct points of $\mathbb{P}^1(\overline{k})$ such that $\hat{h}_\varphi(z_n) \to 0$. Let S_n denote the set of Galois conjugates (over k) of z_n. Then the sequence $\{S_n\}$ is pseudo-equidistributed with respect to $g_{\varphi,v}$ for all $v \in M_k$.*

Proof:

Let N_n be the cardinality of S_n. By Northcott's finiteness theorem, the hypothesis $\hat{h}_\varphi(z_n) \to 0$ (and the fact that the points z_n are all distinct) implies that $N_n \to \infty$ as $n \to \infty$.

For $v \in M_k$ and $n \geq 1$, set

$$g_{v,n} \;:=\; \frac{1}{N_n(N_n-1)} \sum_{\substack{z,w\in S_n \\ z\neq w}} g_{\varphi,v}(z,w) \; .$$

By (4.2), we have

$$g_n := \frac{1}{[k:\mathbb{Q}]} \sum_{v\in M_k} g_{v,n} = \frac{1}{N_n(N_n-1)} \frac{1}{[k:\mathbb{Q}]} \sum_{\substack{z,w\in S_n \\ z\neq w}} \left(\sum_{v\in M_k} g_{\varphi,v}(z,w) \right) = 2\hat{h}_\varphi(z_n) \to 0 \; . \tag{4.10}$$

In particular,

$$\limsup_{n\to\infty} \sum_{v\in M_k} g_{v,n} \;\leq\; 0 \; . \tag{4.11}$$

Now let $t_{v,N} := \inf_{z_1,\ldots,z_N \in \mathbb{P}^1(\mathbb{C}_v)} \frac{1}{N(N-1)} \sum_{i\neq j} g_{\varphi,v}(z_i,z_j)$, so that $\Gamma_{\varphi,v} = \liminf_{N\to\infty} t_{v,N} \geq 0$ by Lemma 3.24. Note that $g_{v,n} \geq t_{v,N_n}$ and $\liminf_{n\to\infty} t_{v,N_n} \geq \liminf_{N\to\infty} t_{v,N}$. Thus

$$\liminf_{n\to\infty} g_{v,n} \;\geq\; 0 \tag{4.12}$$

for all v.

Finally, we apply Lemma 4.5 to $a_n^{(v)} := g_{v,n}$. The hypotheses (F1) and (F2) in that lemma are satisfied because the functions $g_{\varphi,v}$ satisfy (G1) and (G2), and conditions (L1) and (L2)$'$ are satisfied because of (4.12) and (4.11), respectively. We conclude that $\lim_{n\to\infty} g_{v,n} = 0$ for each $v \in M_k$, as desired.

$\square$

As a consequence of this result, and the fact that there are infinitely many pre-periodic points, we obtain

Corollary 4.13. *Let $\varphi \in k(T)$ be a rational function of degree $d \geq 2$. For each place v of k:*
  A) *The constants $\Gamma_{\varphi,v}$ and $I_{\mu_{\varphi,v}}(\mu_{\varphi,v})$ are equal to zero.*
  B) *The Arakelov Green's function $\frac{1}{\log(q_v)} g_{\varphi,v}(z,w)$ is normalized.*

Proof: For A), choose an infinite sequence $\{x_n\}$ of distinct pre-periodic points; then $\hat{h}_\varphi(x_n) = 0$ for each n. Let S_n be the set of Galois conjugates of x_n, and put $N_n = \#(S_n)$. Applying Theorem 4.9, we see that

$$\liminf \frac{1}{N_n(N_n-1)} \sum_{\substack{z,w \in S_n \\ z \neq w}} g_{\varphi,v}(z,w) \; = \; 0.$$

Hence $\Gamma_{\varphi,v}$, defined in (3.23), is ≤ 0. Combined with the inequality $\Gamma_{\varphi,v} \geq 0$ proved in Lemma 3.24, this gives $\Gamma_{\varphi,v} = 0$. Write $\mu = \mu_{\varphi,v}$. Since $D_\infty(\mu) = \Gamma_{\varphi,v}$ by the definitions, it follows from Theorem 3.49 that $I_\mu(\mu) = 0$.

For B), it is only necessary to show that axiom (RS3) (resp (B3)) is satisfied, i.e. we must show that

$$\frac{1}{\log(q_v)} \iint g_{\varphi,v}(z,w)\, d\mu(z) d\mu(w) \; = \; 0 \;.$$

However, this is exactly the assertion that $I_\mu(\mu) = 0$. $\qquad\qquad\square$

4.4. The equidistribution theorem for dynamical systems on $\mathbb{P}^1$.

In this subsection we will show that pseudo-equidistribution, combined with the energy minimization principle, implies equidistribution.

Definition 4.14. If S_n is a finite subset of $\mathbb{P}^1(\mathbb{C}_v)$ for each $n \geq 1$, we say that the sequence $\{S_n\}_{n\geq 1}$ is *equidistributed* with respect to a probability measure μ on $\mathbb{P}^1_{\mathrm{Berk},v}$ over $\mathbb{C}_v$ if the sequence of measures $\delta_n = \delta(S_n)$ converges weakly to μ on $\mathbb{P}^1_{\mathrm{Berk},v}$.

For each $v \in M_k$, pseudo-equidistribution implies equidistribution, in the following precise sense:

Theorem 4.15. *Let $\{S_n\}_{n\geq 1}$ be a sequence of finite subsets of $\mathbb{P}^1(\mathbb{C}_v)$ which is pseudo-equidistributed with respect to $g_{\varphi,v}$. Then $\{S_n\}_{n\geq 1}$ is equidistributed with respect to $\mu_{\varphi,v}$ on $\mathbb{P}^1_{\mathrm{Berk},v}$.*

Proof: Write $\mu = \mu_{\varphi,v}$. Since $\mathbb{P}^1_{\mathrm{Berk},v}$ is compact, it follows from Prohorov's theorem that δ_n has a weakly convergent subsequence. If ν is any weak limit of a subsequence of δ_n, then passing to that subsequence

$$
\begin{aligned}
0 \; &= \; \lim_{n\to\infty} \frac{1}{N_n(N_n-1)} \frac{1}{\log(q_v)} \sum_{\substack{z,w \in S_n \\ z \neq w}} g_{\varphi,v}(z,w) \qquad \text{by pseudo-equidistribution} \\
&= \; \lim_{n\to\infty} \iint_{\mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v} \backslash (\mathrm{Diag})} g_\mu(w,z)\, d\delta_n(w) d\delta_n(z) \\
&\geq \; \iint_{\mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v}} g_\mu(w,z)\, d\nu(w) d\nu(z) \qquad \text{by Lemma 3.46} \\
&= \; I_\mu(\nu) \; \geq \; I_\mu(\mu) \qquad \text{by Theorem 3.45.}
\end{aligned}
$$

Since $I_\mu(\mu) = 0$ by Corollary 4.13, it follows that $I_\mu(\nu) = I_\mu(\mu)$, so Theorem 3.45 gives $\nu = \mu$. $\qquad\qquad\square$

Remark 4.16. For archimedean places v one can give an alternative proof of Theorem 4.15 using a theorem of DeMarco [DeM, Theorem 1.3] instead of Theorem 3.45, and working on $\mathbb{C}^2$ rather than on $\mathbb{P}^1(\mathbb{C})$.

Remark 4.17. For archimedean v, the same proof shows that Theorem 4.15 remains valid if we replace $\mathbb{P}^1_{\mathrm{Berk},v}$ by an arbitrary compact Riemann surface $X/\mathbb{C}$ and $g_{\varphi,v}$ by any normalized Arakelov Green's function on $X(\mathbb{C})$.

Combining Theorem 4.9 and Theorem 4.15, we have finally proved:

Theorem 2.3 (Main Theorem). *Let z_n be a sequence of distinct points of $\mathbb{P}^1(\overline{k})$ such that $\hat{h}_\varphi(z_n) \to 0$. Let S_n denote the set of Galois conjugates (over k) of z_n. Then the sequence $\{S_n\}_{n \geq 1}$ is equidistributed with respect to $\mu_{\varphi,v}$ on $\mathbb{P}^1_{\mathrm{Berk},v}/\mathbb{C}_v$ for all $v \in M_k$.*

5. POTENTIAL THEORY ON RIEMANN SURFACES

The goal of this section is to prove the Energy Minimization Principle (Theorem 5.8) for Arakelov Green's functions on a compact Riemann surface. This result was used in §3, and is needed for the proof of our main equidistribution theorem (Theorem 2.3).

5.1. **Arakelov Green's functions.** Let $X/\mathbb{C}$ be a compact Riemann surface of genus g. Arakelov discovered that by fixing a volume form ω on $X(\mathbb{C})$, one could define an extension of Néron's archimedean local height pairing from divisors of degree zero with disjoint support to arbitrary divisors with disjoint support. When $g \geq 1$, Arakelov defined a canonical volume form ω_{can} (the pullback of the flat metric on the Jacobian of X under an Albanese embedding) which plays a distinguished role in his theory.

The extension of Néron's pairing arises via *Arakelov Green's functions*. We proceed slightly more generally than Arakelov did, using positive measures rather than smooth $(1,1)$-forms.

Recall (Definition 3.35) that a measure μ on $X(\mathbb{C})$ is called *log-continuous* if in every coordinate patch $U \subset X(\mathbb{C})$, the function $\int_U \log|z-w| d\mu(w)$ is finite and continuous for all $z \in U$. For example, any measure μ which locally has the form $\mu = f(z) dx \wedge dy$, where $f(z)$ is continuous and $dx \wedge dy$ is Lebesgue measure, is log-continuous.

We have seen that given an Arakelov Green's function $g_\mu(z,w)$, the associated measure μ is log-continuous. Conversely, given a log-continuous probability measure (i.e., a positive measure of total mass 1) μ on $X(\mathbb{C})$, it follows from [R1, §2.3] or [R2, §4.1] that there is a unique pairing $((z,w))_\mu$ on $X(\mathbb{C}) \times X(\mathbb{C})$ and a corresponding normalized Arakelov Green's function $g_\mu(z,w) : X(\mathbb{C}) \times X(\mathbb{C}) \backslash (\mathrm{Diag}) \to \mathbb{R}$ defined by $g_\mu(z,w) = -\log((z,w))_\mu$ such that axioms (RS1), (RS2) and (RS3) hold.

One way to prove the existence of an Arakelov Green's function $g_\mu(z,w)$ attached to μ is by utilizing a continuously varying family of *canonical distance functions* $[z,w]_\zeta$ on $X(\mathbb{C})$, whose existence is proved in [R1, Theorem 2.1.1]. It is shown in [R1, Theorem 2.3.4] that the integral

$$\int_{X(\mathbb{C})} -\log[z,w]_\zeta \, d\mu(\zeta) \tag{5.1}$$

satisfies properties (RS1) and (RS2) above. Therefore $g_\mu(z,w) := \int_{X(\mathbb{C})} -\log[z,w]_\zeta d\mu(\zeta)$ is an Arakelov Green's function for μ. Furthermore, there is a unique choice of B such that

$\int_{X(\mathbb{C})} -\log[z,w]_\zeta d\mu(\zeta) + B$ satisfies condition (RS3) above and yields a normalized Arakelov Green's function.

Conversely, given an Arakelov Green's function $-\log(\!(z,w)\!)_\mu$, it is shown in [R1, Theorem 2.3.3] that one can construct a continuously varying family of canonical distance functions via

$$[z,w]_\zeta \ := \ \frac{(\!(z,w)\!)_\mu}{(\!(z,\zeta)\!)_\mu (\!(w,\zeta)\!)_\mu}. \tag{5.2}$$

One deduces formula (5.2) from the relation

$$-\log[z,w]_\zeta = -\log[z,w]_p + \log[z,\zeta]_p + \log[w,\zeta]_p + C(p), \tag{5.3}$$

which is valid for all $p, \zeta, z, w \in X(\mathbb{C})$ with $z \neq w$ (see [R1, Corollary 2.1.5]). Here $C(p)$ is a constant depending only on p, and the right-hand side must be suitably interpreted when $z = p$ or $w = p$.

5.2. **Examples of archimedean Arakelov Green's functions.**

Example 5.4. Arakelov Green's functions on $\mathbb{P}^1$

Suppose $X = \mathbb{P}^1$ and $\mu = \mu_{S^1}$ is the uniform probability measure on the unit circle in $\mathbb{C} = \mathbb{P}^1(\mathbb{C}) \backslash \{\infty\}$. Then an Arakelov Green's function associated to μ is

$$g_\mu(z,w) = \begin{cases} -\log|z-w| + \log^+|z| + \log^+|w| & z,w \neq \infty \\ \log^+|z| & w = \infty \\ \log^+|w| & z = \infty. \end{cases} \tag{5.5}$$

Note that the function $g_\mu(z,\infty) = \log^+|z|$ is the Green's function for the unit circle in $\mathbb{C}$ relative to the point at infinity, and is also the archimedean contribution to the logarithmic Weil height on $\overline{\mathbb{Q}} = \mathbb{P}^1(\overline{\mathbb{Q}}) \backslash \{\infty\}$.

If we write (5.5) in terms of a choice of homogeneous coordinates $z = (z_1 : z_2)$, $w = (w_1 : w_2)$, we obtain

$$g_\mu(z,w) = -\log|z \wedge w| + \log||z|| + \log||w||, \tag{5.6}$$

where $z \wedge w = z_1 w_2 - z_2 w_1$ and $||z|| = \max\{|z_1|, |z_2|\}$.

Example 5.7. Arakelov Green's functions on elliptic curves

If $X = E$ is an elliptic curve over $\mathbb{C}$ and $\mu = \mu_{\text{Haar}}$ is the normalized Haar measure on E, then we can take $g_\mu(z,w) = \lambda_\infty(z-w)$, where λ_∞ is a (suitably normalized) archimedean Néron local height function on $E(\mathbb{C})$ (see [Fa, Section 7]). One can explicitly describe the function λ_∞ in terms of the Weierstrass σ-function and the quasi-period homomorphism η (see [Si, Chapter VI]).

5.3. **Statement and discussion of Theorem 5.8.** The main result of this section is the following energy minimization principle for Arakelov Green's functions:

Theorem 5.8. *Let X be a compact Riemann surface, let μ be a log-continuous probability measure on $X(\mathbb{C})$, and let $g_\mu(z,w)$ be an Arakelov Green's function for μ. Define the "energy functional" I_μ on the space $\mathbb{P}$ of probability measures on $X(\mathbb{C})$ by the formula*

$$I_\mu(\nu) \ := \ \iint_{X(\mathbb{C}) \times X(\mathbb{C})} g_\mu(z,w) \, d\nu(z) d\nu(w).$$

Then $I_\mu(\nu) \geq I_\mu(\mu)$ for all probability measures $\nu \in \mathbb{P}$, with equality if and only if $\nu = \mu$.

In other words, μ is the unique probability measure minimizing the energy functional I_μ. Note that by definition, $g_\mu(z,w)$ is normalized if and only if $I_\mu(\mu) = 0$.

The most important difference between Theorem 5.8 and previous energy minimization results on Riemann surfaces (e.g. [R1, Theorem 3.1.12]) is that we consider the space $\mathbb{P}$ of probability measures supported on all of $X(\mathbb{C})$, whereas in classical potential theory, one restricts attention to probability measures supported on a compact set $E \subset X(\mathbb{C})\backslash\{\zeta\}$ for a fixed reference point ζ. For the applications in the present paper, it is crucial to allow ν to vary over all of $\mathbb{P}$, since it is well-known that the canonical measure attached to a rational map can have support equal to all of $\mathbb{P}^1(\mathbb{C})$ (this happens, for example, with the degree 4 Lattès maps associated to multiplication by 2 on an elliptic curve). For polynomial maps, where the filled Julia set stays bounded away from the point at infinity, one can get by with more classical results from potential theory (see [BH]).

We recall that the *capacity* of a compact set $F \subset \mathbb{C}$ is defined as $c(F) = e^{-V(F)}$, where $V(F)$ (the "Robin's constant" of F) is the infimum (which may be a real number or $+\infty$) over all probability measures ν supported on F of expression (5.10) below. Theorem 5.8 can be viewed as a generalization of the following fundamental result from capacity theory (see [Ran]):

Theorem 5.9. *Let F be a compact subset of $\mathbb{C}$ having positive capacity. Then there exists a unique probability measure μ_F supported on F (called the equilibrium measure for F) which minimizes the energy functional*

$$I(\nu) \;=\; \iint_{F \times F} -\log|z-w|\, d\nu(z)d\nu(w). \tag{5.10}$$

We claim that Theorem 5.8 implies Theorem 5.9 for all compact sets F such that each $x \in F$ is regular for the Dirichlet problem. (For example, this holds if each connected component of F is a continuuum; see [Ts], Theorem I.11, p.7). Such a set necessarily has positive capacity ([Ts], Theorem III.5, p.56). If $g_F(z)$ is a Green's function for $\mathbb{C}\backslash F$ relative to ∞, then $g_F(z)$ is continuous ([Ts], Theorem III.36, p.82). One verifies easily that if $\mu_F = -\Delta g_F(z)$, then the function

$$-\log|z-w| + g_F(z) + g_F(w)$$

satisfies conditions (RS1) and (RS2) above, and therefore the normalized Arakelov Green's function for μ_F is given for $z,w \in \mathbb{C}, z \neq w$ by

$$g_{\mu_F}(z,w) \;=\; -\log|z-w| + g_F(z) + g_F(w) - C,$$

where the constant C is chosen so that (RS3) is satisfied. (Compare with Example 5.4). Since μ_F is supported on F and $g_F \equiv 0$ on F, we must in fact have

$$C \;=\; \iint_{F \times F} -\log|z-w|\, d\mu_F(z)d\mu_F(w).$$

Also, if ν is any probability measure supported on F, then Theorem 5.8 implies that

$$I_{\mu_F}(\nu) \;=\; \iint_{F \times F} -\log|z-w|\, d\nu(z)d\nu(w) - C \;\geq\; 0.$$

Therefore μ_F is the unique probability measure supported on F which minimizes the energy functional I (i.e., μ_F is the equilibrium measure for F). By the definition of capacity, we

also see that the constant C is just the Robin's constant $V(F)$ of F, so that

$$g_{\mu_F}(z, w) \; = \; -\log|z - w| + g_F(z) + g_F(w) + \log c(F).$$

Remark 5.11. P. Autissier has obtained essentially the same result as Theorem 5.8 in the case where μ is the equilibrium measure of a compact set $F \subset \mathbb{C}$.

We have already discussed the relationship between Theorem 5.8 and classical potential theory in $\mathbb{C}$. There is also a relationship between Theorem 5.8 and a result which is used in Arakelov theory in order to establish an analogue of the Riemann-Roch theorem for arithmetic surfaces (see [La]). Indeed, when $g \geq 1$ and $\mu = \omega_{\mathrm{can}}$, the nonnegativity of $I_\mu(\nu)$ in Theorem 5.8 is the continuous analogue of the following theorem of Faltings [Fa], originally proved using the spectral theory of the Laplacian:

Theorem 5.12 (Faltings). *For each integer $n \geq 1$, choose an n-tuple $z_1, \dots, z_n$ of distinct points in $X(\mathbb{C})$. Then*

$$\liminf_{n \to \infty} \frac{1}{n(n-1)} \sum_{i \neq j} g_{\omega_{\mathrm{can}}}(z_i, z_j) \; \geq \; 0 \; .$$

It is easy to see that Theorem 5.8 implies Theorem 5.12. Indeed, if δ_n is the discrete probability measure supported equally at $z_1, \dots, z_n$ and if ν is any weak limit of a subsequence of δ_n, then

$$\frac{1}{n(n-1)} \sum_{i \neq j} g_{\omega_{\mathrm{can}}}(z_i, z_j) \; = \; \iint_{X(\mathbb{C}) \times X(\mathbb{C}) \backslash (\mathrm{Diag})} g_{\omega_{\mathrm{can}}}(z, w) \, d\delta_n(z) d\delta_n(w),$$

and it follows from Lemma 3.46 that

$$\liminf_{n \to \infty} \iint_{X(\mathbb{C}) \times X(\mathbb{C}) \backslash (\mathrm{Diag})} g_{\omega_{\mathrm{can}}}(z, w) \, d\delta_n(z) d\delta_n(w) \geq \iint_{X(\mathbb{C}) \times X(\mathbb{C})} g_{\omega_{\mathrm{can}}}(z, w) \, d\nu(z) d\nu(w).$$

This last quantity is nonnegative by Theorem 5.8, proving the claim.

5.4. **Proof of Theorem 5.8.** We now turn to the proof of Theorem 5.8. The proof uses the representation of $g_\mu(z, w)$ in terms of the canonical distance function, and is similar to the classical proof of Theorem 5.9. Namely, our plan is to prove analogues of Maria's theorem and Frostman's theorem, and to deduce Theorem 5.8 from those results. As discussed in §5.3, a key difference between Theorem 5.8 and Theorem 5.9 is the presence in Theorem 5.9 of a fixed reference point at infinity. It is the assumption that μ is log-continuous which ultimately allows us to apply techniques from classical potential theory to the present situation.

We fix a (possibly non-normalized) Arakelov Green's function $g_\mu(z, w)$ for μ, and for each $\zeta \in X(\mathbb{C})$ we define a canonical distance function $[z, w]_\zeta$ by (5.2).

Using formula (5.3), we see that if $\zeta \in X(\mathbb{C})$ then for all $z, w \neq \zeta$

$$\begin{aligned} g_\mu(z, w) \; &= \; \int -\log[z, w]_p \, d\mu(p) \\ &= \; -\log[z, w]_\zeta - u_\mu(z, \zeta) - u_\mu(w, \zeta) + C_\zeta \; , \end{aligned} \tag{5.13}$$

where

$$u_\mu(z, \zeta) := \int_{X(\mathbb{C})} -\log[z, w]_\zeta \, d\mu(w) \; .$$

By Proposition 2.1.3 of [R1], for fixed $\zeta \in X(\mathbb{C})$, the two-variable function $-\log[z, w]_\zeta$ can be expressed locally on $X(\mathbb{C}) \times X(\mathbb{C})$ as a linear combination of $\log|z - w|, \log|z - \zeta|$,

$\log |w - \zeta|$, and a continuous function which is harmonic in z and w separately. Since μ is log-continuous, it follows that for fixed ζ, the function $u_\mu(z, \zeta)$ is continuous on $X(\mathbb{C}) \backslash \{\zeta\}$.

If ν is any probability measure on $X(\mathbb{C})$, we define the generalized potential function $u_\mu(z, \nu) : X(\mathbb{C}) \to \mathbb{R} \cup \{\infty\}$ to be

$$u_\mu(z, \nu) := \int_{X(\mathbb{C})} g_\mu(z, w) \, d\nu(w) .$$

Lemma 5.14. *For any probability measure ν on $X(\mathbb{C})$, we have $\Delta u_\nu(z, \mu) = \nu - \mu$ as distributions.*

Proof: For any test function ψ, we have

$$\int \psi(z) \Delta u_\nu(z, \mu) = \int u_\nu(z, \mu) \Delta \psi(z)$$

$$= \int \left(\int g_\mu(z, w) \, d\nu(w) \right) \Delta \psi(z)$$

$$= \int \left(\int g_\mu(z, w) \, \Delta \psi(z) \right) d\nu(w) \qquad \text{(Fubini's theorem)}$$

$$= \int \left(\int \psi(z) \Delta g_\mu(z, w) \right) d\nu(w)$$

$$= \int \psi(w) - \left(\int \psi(z) \, d\mu(z) \right) d\nu(w)$$

$$= \int \psi(w) d\nu(w) - \int \psi(z) d\mu(z).$$

The use of Fubini's theorem at the third step is justified by the fact that the measure $\Delta \psi(z)$ locally has the form $f(z) \, dx \wedge dy$ for a continuous function $f(z)$. It follows that positive and negative parts $\Delta \psi^+$ and $\Delta \psi^-$ in the Jordan decomposition of $\Delta \psi$ are log-continuous. Now Fubini's theorem, in the form given in ([Rud], Theorem 7.8(b), p.150), says that $\int (\int g_\mu(z, w) \, \Delta \psi^\pm(z)) \, d\nu(w) = \int (\int g_\mu(z, w) \, d\nu(w)) \Delta \psi^\pm(z)$. $\qquad \square$

Lemma 5.15. *For any probability measure ν on $X(\mathbb{C})$,*
   A) $u_\nu(z, \mu)$ is a lower semicontinuous function on $X(\mathbb{C})$.
   B) $u_\nu(z, \mu)$ is continuous and subharmonic outside $\text{supp}(\nu)$.

Proof: For each $M \in \mathbb{R}$ define $g_\mu^{(M)}(z, w) := \min\{M, g_\mu(z, w)\}$, with $g_\mu^{(M)}(z, z) := M$ for all z. Then $g_\mu^{(M)}(z, w)$ is a continuous function on $X(\mathbb{C}) \times X(\mathbb{C})$, and

$$u_\nu(z, \mu) = \lim_{M \to \infty} \int_{X(\mathbb{C})} g_\mu^{(M)}(z, w) d\nu(w).$$

Therefore $u_\nu(z, \mu)$ is lower semicontinuous, being an increasing limit of continuous functions. This proves part A) of the lemma.

To prove part B), note that if $z_0 \notin \text{supp}(\nu)$, then by definition there exists an open neighborhood U of z_0 whose closure is disjoint from $\text{supp}(\nu)$, such that $\nu(U) = 0$. For $z \in U$, we have

$$u_\nu(z, \mu) = \int_{X(\mathbb{C})} g_\mu(z, w) d\nu(w) = \int_{X(\mathbb{C}) \backslash U} g_\mu(z, w) d\nu(w).$$

Since $g_\mu(z, w)$ is uniformly continuous on $\overline{U} \times (X(\mathbb{C}) \backslash U)$, it follows that $u_\nu(z, \mu)$ is continuous on U, and in particular at z_0.

The fact that $u_\nu(z, \mu)$ is subharmonic outside $\mathrm{supp}(\nu)$ follows from the fact that $\Delta u_\nu(z, \mu) = \nu - \mu$, and thus $-\Delta u_\nu(z, \mu) = \mu$ is a positive distribution on the complement of $\mathrm{supp}(\nu)$. (See [Kl], Theorem 2.9.11, p.67). $\qquad\square$

The potential function $u_\nu(z, \mu)$ has additional continuity properties besides those given by Lemma 5.15. For example, we have the following (compare with [Ran, Theorem 3.1.3]).

Proposition 5.16. *Let ν be a probability measure on $X(\mathbb{C})$. Then for every $z_0 \in K := \mathrm{supp}(\nu)$, we have*

$$\limsup_{z \to z_0} u_\nu(z, \mu) \;=\; \limsup_{\substack{z \to z_0 \\ z \in K}} u_\nu(z, \mu).$$

Proof: Let U be the complement in $X(\mathbb{C})$ of K. Without loss of generality, we may assume that $U \neq \emptyset$ and that $z_0 \in \partial U$. It is easy to see that the desired result is then equivalent to

$$\limsup_{\substack{z \to z_0 \\ z \notin K}} u_\nu(z, \mu) \;\leq\; \limsup_{\substack{z' \to z_0 \\ z \in K}} u_\nu(z', \mu) .$$

If $u_\nu(z_0, \mu) = \infty$ then by lower semicontinuity we have $\lim_{z \to z_0} u_\nu(z, \mu) = \infty$ and the result is trivial. Therefore we may assume that $u_\nu(z_0, \mu) < \infty$, in which case $\nu(\{z_0\}) = 0$. It follows that given $\epsilon > 0$, there exists a closed disk D centered at z_0 such that $\nu(D) < \epsilon$.

Let $F := D \cap K$ (so that $z_0 \in F$), and fix $z \in D \backslash F$.

Claim: There exists a constant $C > 0$ (independent of z) and a point $z' \in F$ (depending on z) such that

$$g_\mu(z, w) \;\leq\; g_\mu(z', w) + C \tag{5.17}$$

for all $w \in F$.

Before proving the claim, let's see how it gives what we want. Integrating both sides over F against ν, we get

$$
\begin{aligned}
\int_F g_\mu(z, w) d\nu(w) &\leq \int_F g_\mu(z', w) d\nu(w) + C \cdot \nu(F) \\
&\leq \int_K g_\mu(z', w) d\nu(w) - \int_{K \backslash F} g_\mu(z', w) d\nu(w) + C \cdot \epsilon \\
&= u_\nu(z', \mu) - \int_{K \backslash F} g_\mu(z', w) d\nu(w) + C \cdot \epsilon.
\end{aligned}
$$

Therefore

$$\int_F g_\mu(z, w) d\nu(w) + \int_{K \backslash F} g_\mu(z', w) d\nu(w) \;\leq\; u_\nu(z', \mu) + C \cdot \epsilon. \tag{5.18}$$

As $z \to z_0$ in $D \backslash F$, we have $z' \to z_0$ in F also (take $w = z_0$ in (5.17)). By the continuity of $g_\mu(z, w)$ on $D \times (K \backslash F)$, as $z' \to z_0$ and $z \to z_0$ we have

$$\int_{K \backslash F} \left(g_\mu(z', w) - g_\mu(z, w) \right) d\nu(w) \to 0.$$

Therefore (5.18) gives

$$\limsup_{\substack{z \to z_0 \\ z \notin K}} \int_K g_\mu(z, w)\, d\nu(w) \;\leq\; \limsup_{\substack{z' \to z_0 \\ z \in K}} u_\nu(z', \mu) + C \cdot \epsilon.$$

As $\epsilon > 0$ was arbitrary, this gives the desired result.

It remains to prove the claim. Choose $\zeta \in X(\mathbb{C})$, a small disk D_ζ around ζ with $D_\zeta \cap D = \emptyset$, and a constant $M_\zeta \in \mathbb{R}$ so that $|g_\mu(z, \zeta)| \leq M_\zeta$ for all $z \notin D_\zeta$. By [R1, proof of Theorem 3.1.6], there is a constant C' (depending only on ζ and D) such that

$$- \log[z, w]_\zeta \ \leq \ - \log[z', w]_\zeta + C'$$

for all $z, z' \in D$ with $z \neq z'$.

Using formula (5.2), we see that

$$g_\mu(z, w) - g_\mu(z', w) - g_\mu(z, \zeta) + g_\mu(z', \zeta) \ \leq \ C'$$

and therefore

$$g_\mu(z, w) - g_\mu(z', w) \ \leq \ C' + 2M_\zeta,$$

which proves the claim. $\qquad \square$

We thus obtain the following analogue of Maria's theorem (see [R1, Theorem 3.1.6]):

Corollary 5.19. *If M is a real number such that $u_\nu(z, \mu) \leq M$ on $\mathrm{supp}(\nu)$, then $u_\nu(z, \mu) \leq M$ on all of $X(\mathbb{C})$.*

Proof: We may clearly assume that the complement U of $\mathrm{supp}(\nu)$ is non-empty. By Lemma 5.15, u is subharmonic on U, and by Proposition 5.16, for each boundary point z_0 of $\mathrm{supp}(\nu)$ we have

$$\limsup_{\substack{z \to z_0 \\ z \in U}} u_\nu(z, \mu) = \limsup_{\substack{z \to z_0 \\ z \in \mathrm{supp}(\nu)}} u_\nu(z, \mu) \leq M.$$

The result now follows from the maximum principle for subharmonic functions (applied to each connected component of U). $\qquad \square$

Next we have the following result, proved by a standard argument:

Lemma 5.20. *There exists an energy-minimizing measure ν_0 for the functional I_μ.*

Proof: Let $V_\mu := \inf_{\nu \in \mathbb{P}} I_\mu(\nu)$, and choose a sequence of probability measures μ_n in $\mathbb{P}$ such that $\lim_{n \to \infty} I_\mu(\mu_n) = V_\mu$. Passing to a subsequence if necessary, we may assume that μ_n converges weakly to some measure ν_0. We claim that

$$V_\mu \ = \ \liminf_{n \to \infty} I_\mu(\mu_n) \ \geq \ I_\mu(\nu_0) \ . \tag{5.21}$$

Given this claim, we see that since $V_\mu = \inf_{\nu \in \mathbb{P}} I_\mu(\nu)$, we must have $I_\mu(\nu_0) = V_\mu$.

To prove (5.21), we proceed as in the proof of Lemma 3.46:

$$\liminf_{n \to \infty} \iint_{X(\mathbb{C}) \times X(\mathbb{C})} g_\mu(z, w) \, d\mu_n(z) d\mu_n(w)$$

$$\geq \lim_{M \to \infty} \liminf_{n \to \infty} \iint \min\{M, g_\mu(z, w)\} \, d\mu_n(z) d\mu_n(w) \qquad (\text{since } (*) \geq \min\{M, (*)\})$$

$$= \lim_{M \to \infty} \iint \min\{M, g_\mu(z, w)\} \, d\nu_0(z) d\nu_0(w) \qquad (\mu_n \to \nu_0 \text{ weakly})$$

$$= \iint g_\mu(z, w) \, d\nu_0(z) d\nu_0(w) \qquad (\text{monotone convergence theorem}).$$

$\qquad \square$

Recall from [R1, §3.1] that if $E \subset X(\mathbb{C})$ is compact and $\zeta \notin E$, then the capacity $\gamma_\zeta(E)$ of E (with respect to ζ) is defined to be $\gamma_\zeta(E) := e^{-V_\zeta(E)}$, where

$$V_\zeta(E) \ := \ \inf_{\nu \in \mathbb{P}(E)} \iint -\log[z,w]_\zeta \, d\nu(z)d\nu(w)$$

and $\mathbb{P}(E)$ is the set of probability measures supported on E.

Similarly, for any compact E we define the μ-capacity of E by $\gamma_\mu(E) := e^{-V_\mu(E)}$, where

$$V_\mu(E) \ := \ \inf_{\nu \in \mathbb{P}(E)} \iint g_\mu(z,w) \, d\nu(z)d\nu(w) \ .$$

Note that if $E = X(\mathbb{C})$ then $I_\mu(\mu) = V_\mu(E)$.

Lemma 5.22. *If $E \subset X(\mathbb{C})$ is compact and $\zeta \notin E$, then $\gamma_\mu(E) = 0$ if and only if $\gamma_\zeta(E) = 0$.*

Proof: This follows immediately from formula (5.2), which implies that for each $\nu \in \mathbb{P}(E)$,

$$\begin{aligned}
V_\zeta(E) \ &= \ \iint -\log[z,w]_\zeta \, d\nu(z)d\nu(w) \\
&= \ \iint (g_\mu(z,w) - g_\mu(z,\zeta) - g_\mu(w,\zeta)) \, d\nu(z)d\nu(w) \\
&= \ V_\mu(E) - 2 \int_E g_\mu(z,\zeta) \, d\nu(z).
\end{aligned}$$

Here $\int_E g_\mu(z,\zeta) \, d\nu(z) < \infty$, since $\mathrm{supp}(\nu) \subseteq E$ and $\zeta \notin E$. $\qquad\square$

Remark 5.23. If $\zeta, \zeta' \notin E$ then it follows from Lemma 5.22 that $\gamma_\zeta(E) = 0$ if and only if $\gamma_{\zeta'}(E) = 0$. In particular, it makes sense to speak of a set of capacity zero on $X(\mathbb{C})$ without reference to a particular base point ζ.

Lemma 5.24. *Let ν be a probability measure on $X(\mathbb{C})$ such that $I_\mu(\nu) < \infty$. Then $\nu(A) = 0$ for any Borel subset $A \subset X(\mathbb{C})$ of capacity zero.*

Proof: Suppose to the contrary that $\nu(A) > 0$. Then for some compact subset $A' \subseteq A$ we also have $\nu(A') > 0$, so without loss of generality we may assume that A itself is compact. Recall that there exists $M \in \mathbb{R}$ such that $g_\mu(z,w) \geq -M$ for all $z,w \in X(\mathbb{C})$. Then

$$\int_A \int_A (g_\mu(z,w) + M) \, d\nu(z)d\nu(w) \ \leq \ \int_{X(\mathbb{C})} \int_{X(\mathbb{C})} (g_\mu(z,w) + M) \, d\nu(z)d\nu(w),$$

so that

$$\begin{aligned}
\int_A \int_A g_\mu(z,w) \, d\nu(z)d\nu(w) \ &\leq \ \int_{X(\mathbb{C})} \int_{X(\mathbb{C})} g_\mu(z,w) \, d\nu(z)d\nu(w) + M \cdot (1 - \nu(A)^2) \\
&\leq \ I_\mu(\nu) + M \ < \ \infty.
\end{aligned}$$

Define a probability measure ν' on A by setting $\nu' := \frac{1}{\nu(A)}\nu|_A$. Then

$$I_\mu(\nu') \ \leq \ \frac{1}{\nu(A)^2} (M + I_\mu(\nu)) \ < \ \infty \ ,$$

so that $\gamma_\mu(A) > 0$, a contradiction. $\qquad\square$

The following result is an analogue of Frostman's theorem (see [R1, Theorem 3.1.7]):

Theorem 5.25. *Let ν_0 be any probability measure which minimizes the functional I_μ, and let $V := I_\mu(\nu_0)$. Then the potential function $u(z) := u_{\nu_0}(z, \mu)$ on $X(\mathbb{C})$ satisfies:*

A) $u(z) = V$ for all $z \in X(\mathbb{C})$ outside a set of capacity zero.

B) $u(z) \leq V$ for all $z \in X(\mathbb{C})$.

Proof: We first show that $u(z) \geq V$ for all $z \in X(\mathbb{C})$ outside a set of capacity zero. For each $n \geq 1$, put $A_n := \{z \in X(\mathbb{C}) : u(z) \leq V - \frac{1}{n}\}$. The lower semicontinuity of $u_{\nu_0}(z)$ shows that each A_n is closed, and we have $\cup A_n = A := \{z \in X(\mathbb{C}) : u(z) < V\}$. Clearly $A \neq X(\mathbb{C})$, since $\int u(z)d\nu_0(z) = V$ and ν_0 is a positive measure.

If ζ is any point in the complement of A, we claim that $\gamma_\zeta(A) = 0$. By Lemma 5.22, it suffices to show that $\gamma_\mu(A) = 0$. Suppose on the contrary that $\gamma_\mu(A) > 0$. To obtain a contradiction, we first construct disjoint closed subsets E_1, E_2 of $X(\mathbb{C})$ as follows.

By [R1, Proposition 3.1.5], we must have $\gamma_\zeta(A_n) > 0$, and hence $\gamma_\mu(A_n) > 0$, for some n. Thus for a suitable $\epsilon > 0$ and $n \geq 1$, we have $u(z) < V - 2\epsilon$ on $E_1 := A_n$ and $\gamma_\mu(E_1) > 0$.

As $\int u(z)d\nu_0(z) = V$, there exists $z_0 \in \operatorname{supp}(\nu_0)$ such that $u_\nu(z_0, \mu) > V - \epsilon$. Lower semicontinuity implies that this inequality remains valid in a closed disk D around z_0, which we may assume to be disjoint from E_1. Since $z_0 \in \operatorname{supp}(\nu_0)$, we have $\nu_0(D) > 0$. Let $E_2 := D$, and let $W := \nu_0(E_2) > 0$.

Since $\gamma_\mu(E_1) > 0$, there exists a probability measure ν' supported on E_1 such that $I_\mu(\nu') < \infty$. Define a new measure σ on $X(\mathbb{C})$ by setting

$$\sigma := \begin{cases} W\nu' & \text{on } E_1 \\ -\nu_0 & \text{on } E_2 \\ 0 & \text{elsewhere} \end{cases}$$

Then $\sigma(E_1) = W$, $\sigma(E_2) = -W$, and $\sigma(X(\mathbb{C})) = 0$. Note that for each real number $t \in [0,1]$, $\nu_0 + t\sigma$ is a probability measure on $X(\mathbb{C})$. As in [R1, proof of Theorem 3.1.6], we calculate that $I_\mu(\sigma) < \infty$ and

$$I_\mu(\nu_0 + t\sigma) - I_\mu(\nu_0) \leq (-2W\epsilon) \cdot t + I_\mu(\sigma) \cdot t^2. \tag{5.26}$$

For t sufficiently small, the right-hand side of (5.26) is negative, contradicting the fact that $I_\mu(\nu_0) = V$ is the minimum possible energy of a probability measure on $X(\mathbb{C})$. This contradiction proves that A has capacity zero, and by construction we have $u(z) \geq V$ for all $z \in X(\mathbb{C}) \backslash A$.

Next, we show that $u(z) \leq V$ on $\operatorname{supp}(\nu_0)$. To see this, suppose for the sake of contradiction that $u_\nu(z_0, \mu) > V$ for some $z_0 \in \operatorname{supp}(\nu_0)$. By the lower semicontinuity of u, there exists $\epsilon > 0$ and a closed disk D around z_0 such that $u(z) > V + \epsilon$ on D.

As $z_0 \in \operatorname{supp}(\nu_0)$, the number $T := \nu_0(D)$ is positive.

We have already seen that $u(z) \geq V$ for all $z \in X(\mathbb{C})$, except on a set A of capacity (and hence, by Lemma 5.24, of ν_0-measure) zero. Therefore

$$V = \int u(z)d\nu_0(z) \geq V(1 - T) + (V + \epsilon)T > V,$$

a contradiction. It follows that $u(z) \leq V$ on $\operatorname{supp}(\nu_0)$ as claimed.

Finally, Corollary 5.19 now shows that $u(z) \leq V$ on all of $X(\mathbb{C})$, which proves both A) and B). $\qquad \square$

We can now prove Theorem 5.8.

Proof: As in the statement of Theorem 5.25, let ν_0 be any probability measure which minimizes the functional $I_\mu(\nu)$, and let $V = I_\mu(\nu_0)$. Since $I_\mu(\mu) < \infty$ by (3.33), we must have $V < \infty$ as well.

Integrating over $X(\mathbb{C}) \times X(\mathbb{C})$ and applying Fubini's theorem, we have

$$\int \left(\int g_\mu(z,w) \, d\mu(z) \right) d\nu_0(w) = \int \left(\int g_\mu(z,w) d\nu_0(w) \right) d\mu(z). \tag{5.27}$$

The interchange of order of integration is justified by the same reasons as in Lemma 5.14, because μ is log-continuous.

Here the left side equals $I_\mu(\mu)$, since $\int g_\mu(z,w) d\mu(z)$ is constant by property (RS3)$'$ of Arakelov Green's functions.

On the other hand, Theorem 5.25 shows that $u(z) := \int g_\mu(z,w) \, d\nu_0(w) = V$ outside a set A with capacity zero. Since $I_\mu(\mu) < \infty$ by assumption, Lemma 5.24 shows that $\mu(A) = 0$. Hence the right side of (5.27) is V.

Combining these gives $I_\mu(\mu) = V$. Therefore μ is also energy-minimizing.

To see that $\mu = \nu_0$, note first that sets of capacity zero have Lebesgue measure zero in any coordinate patch on $X(\mathbb{C})$. (This follows from Lemma 5.24). Hence $u(z) = V$ almost everywhere with respect to Lebesgue measure, and consequently $\Delta u(z) = 0$. On the other hand, Lemma 5.14 gives us the distributional identity $\Delta u(z) = \nu_0 - \mu$. Therefore $\mu = \nu_0$ as desired. $\qquad \square$

6. Comparison and calculation of various capacities

As before, we let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$ defined over a number field k, and let $F = (F_1, F_2) : \mathbb{A}^2 \to \mathbb{A}^2$ be a lifting of φ, where $F_1(x,y)$ and $F_2(x,y)$ are homogeneous polynomials of degree d with coefficients in k having no common factors over $\overline{k}$.

In this section we will prove Theorem 3.16, the resultant formula for the homogeneous transfinite diameter of the filled Julia set $K_{F,v}$:

$$d_\infty^0(K_{F,v}) = |\mathrm{Res}(F)|_v^{-1/d(d-1)} . \tag{6.1}$$

We do this by considering various notions of capacity: the local and global sectional capacities and the Chebyshev constant studied in [RL] and [RLV], and the homogenous sectional capacity and homogeneous transfinite diameter, which are introduced here for the first time. The reason for this proliferation of capacities is that we can compute the sectional capacity, and there are standard methods for proving inequalities between various other types of capacities. In outline, the plan is to first prove

$$\text{local sectional capacity} = |\mathrm{Res}(F)|_v^{-1/d(d-1)}$$

by proving an upper bound for the local sectional capacity and using the fact that the global sectional capacity is the product of the local sectional capacities, and then to show that for circled sets,

$$\text{local sectional capacity} = \text{homogeneous sectional capacity}$$
$$= \text{homogenous transfinite diameter}.$$

We prove more in this section than is strictly needed for our application to dynamics. It is our hope that the ideas developed here will be useful in other contexts as well. In particular,

it would be interesting to know if the resultant formula for the local sectional capacity of the pullback of a ball (Proposition 6.4) generalizes to higher dimensions.

6.1. **Sectional capacities of polynomial domains.** We can view F as defining a finite map $\tilde{F} : \mathbb{P}^2 \to \mathbb{P}^2$ given in homogeneous coordinates by $\tilde{F}(X : Y : Z) = (F_1(X, Y) : F_2(X, Y) : Z^d)$. Its action on the affine patch $\mathbb{A}^2$ is given by F and it stabilizes hyperplane $H = \{Z = 0\}$, which we identify with $\mathbb{P}^1$, where its action is given by φ. The map $\tilde{F}$ has degree d^2, and $\tilde{F}^*(H) = d \cdot H$.

The definition of the sectional capacity for sets in $\mathbb{P}^2$, relative to the divisor H, is as follows.

For each place v of k, let $E_v \subset \mathbb{P}^2(\mathbb{C}_v)$ be a nonempty set which is stable under the group of continuous automorphisms $\mathrm{Gal}^c(\mathbb{C}_v/k_v) \cong \mathrm{Gal}(\tilde{k}_v/k_v)$ and is bounded away from $H(\mathbb{C}_v)$ under the v-adic metric on $\mathbb{P}^2(\mathbb{C}_v)$. For all but finitely many v we assume that $E_v = B(0,1) \times B(0,1) \subset \mathbb{A}^2(\mathbb{C}_v)$, the 'trivial set' for v with respect to H. We will call these assumptions the *Standard Hypotheses*.

Put $\mathbb{E} = \prod_v E_v \subset \mathbb{A}_k$, where $\mathbb{A}_k$ is the adele ring of k.

For each $n \geq 0$, identify the space of sections $\Gamma(n) := H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))$ with the set of homogeneous polynomials in $k[X, Y, Z]$ of degree n. Consider the basis for $k[X, Y, Z]$ given by the monomials $\{X^k Y^\ell Z^m\}$; equip it with the term order $\prec$ given by the lexicographic order with $Z \prec X \prec Y$, graded by the degree. We call this structure the 'monic basis'; it is the key ingredient used in defining local sectional capacities. (Any other term order graded by the degree would work; this one is most directly compatible with dehomogenization.)

For each place v of k, let vol_v be additive Haar measure on k_v (normalized so that $\mathrm{vol}_v(\mathcal{O}_v) = 1$ if v is nonarchimedean, and given by Lebesgue measure on $\mathbb{R}$ or $\mathbb{C}$ if v is archimedean). Let $\mathrm{vol}_{\mathbb{A}}$ be the additive Haar measure on the adele ring $\mathbb{A}_k$ given by the product of the measures vol_v. For each n, by transport of structure using the monic basis we obtain Haar measures vol_v on the vector spaces $\Gamma_{\varphi,v}(n) = k_v \otimes_k \Gamma(n)$ and $\mathrm{vol}_{\mathbb{A}}$ on the $\mathbb{A}_k$-module $\Gamma_{\mathbb{A}}(n) = \mathbb{A}_k \otimes_k \Gamma(n)$.

To define norms, we dehomogenize at Z, writing $x = X/Z$, $y = Y/Z$, and identify $\Gamma(n)$ with the space of polynomials in $k[x, y]$ of total degree $\leq n$. We view these as functions on $\mathbb{A}^2$. Put

$$\begin{aligned}
\mathcal{F}_v(n) &= \{f \in \Gamma_{\varphi,v}(n) : \|f\|_{E_v} \leq 1\} , \\
\mathcal{F}_{\mathbb{A}}(n) &= \left(\prod_v \mathcal{F}_v(n)\right) \cap \Gamma_{\mathbb{A}}(n) .
\end{aligned}$$

The local sectional capacity $S_\gamma(E_v, H)$ is defined by

$$-\log(S_\gamma(E_v, H)) = \lim_{n\to\infty} \frac{3!}{n^3} \log(\mathrm{vol}_v(\mathcal{F}_v(n)))$$

and the global sectional capacity $S_\gamma(\mathbb{E}, H)$ by

$$-\log(S_\gamma(\mathbb{E}, H)) = \lim_{n\to\infty} \frac{3!}{n^3} \log(\mathrm{vol}_{\mathbb{A}}(\mathcal{F}_{\mathbb{A}}(n))) .$$

In [RL] it is shown that under the Standard Hypotheses, the limits defining $S_\gamma(E_v, H)$ and $S_\gamma(\mathbb{E}, H)$ exist, and that

$$S_\gamma(\mathbb{E}, H) \;=\; \prod_v S_\gamma(E_v, H) \,.$$

We now apply this to polydiscs in $\mathbb{C}_v^2$ and their pullbacks by F. Given $z = (x, y) \in \mathbb{C}_v^2$, write $\|z\|_v = \max(|x|_v, |y|_v)$. (For archimedean v, this is a different definition of $\|z\|_v$ than we used in §3.)

For each $R_v > 0$, put

$$B_v(R_v) \;:=\; B(0, R_v)^2 \;=\; \{z \in \mathbb{C}_v^2 : \max(|x|_v, |y|_v) \leq R_v\} \,.$$

Thus

$$F^{-1}(B_v(R_v)) \;=\; \{z \in \mathbb{C}_v : \max(|F_1(z)|_v, |F_2(z)|_v) \leq R_v\} \,.$$

Given a collection of numbers $\vec{R} = \{R_v\}$ with $R_v = 1$ for all but finitely many v, define the adelic sets

$$\mathbb{B}(\vec{R}) \;=\; \prod_v B_v(R_v) \,,$$

$$F^{-1}(\mathbb{B}(\vec{R})) \;=\; \prod_v F^{-1}(B_v(R_v)) \,.$$

Proposition 6.2.
   A) *For each v, the local sectional capacity $S_\gamma(B_v(R_v), H)$ equals R_v^2.*
   B) *The global sectional capacity $S_\gamma(\mathbb{B}(\vec{R}), H)$ equals $\prod_v R_v^2$.*
   C) *The global sectional capacity $S_\gamma(F^{-1}(\mathbb{B}(\vec{R})), H)$ equals $(\prod_v R_v^2)^{1/d}$.*

Proof: Part A) follows from the fact that the logarithmic capacity of a ball in $\mathbb{P}^1$ is $\gamma_\infty(B(0, R_v)) = R_v$ (see, e.g. [R1], Example 5.2.15, p.352), together with the formula for the the sectional capacity of a product set $E_v = E_{v,1} \times E_{v,2} \subset \mathbb{C}_v^2$:

$$S_\gamma(E_v, H) = \gamma_\infty(E_{v,1}) \cdot \gamma_\infty(E_{v,2}) \tag{6.3}$$

(see [RL], Example 4.3, p.558).

Part B) follows from part A) and ([RL], Theorem 3.1, p.552).

Part C) follows from part B) and functorial properties of the global sectional capacity. By the the pullback formula for finite maps ([RLV], Theorem 10.1, p.54), we have

$$S_\gamma(F^{-1}(\mathbb{B}(\vec{R})), dH) \;=\; S_\gamma(\mathbb{B}(\vec{R}), H)^{d^2} \,,$$

since $\tilde{F}^{-1}(\mathbb{B}(\vec{R})) = F^{-1}(\mathbb{B}(\vec{R}))$, $\tilde{F}^*(H) = dH$, and $\deg(\tilde{F}) = d^2$. Also, by the homogeneity of the sectional capacity in its second variable ([RLV], Theorem C (5), p.9), $S_\gamma(\mathbb{E}, dH) = S_\gamma(\mathbb{E}, H)^{d^3}$ for any $\mathbb{E}$ in $\mathbb{P}^2$. Combining these gives C).   □

Determining the local sectional capacity $S_\gamma(F^{-1}(B_v(R_v)), H)$ is more difficult. As before, let $\mathrm{Res}(F)$ denote the resultant of F_1 and F_2.

Proposition 6.4. *For each v,*

$$S_\gamma(F^{-1}(B_v(R_v)), H) \;=\; (R_v^2)^{1/d} \cdot |\mathrm{Res}(F)|_v^{-1/d^2} \,.$$

Before giving the proof, we will need a lemma. For each m, write $\Gamma_v^0(m)$ for the space of homogeneous polynomials in $k_v[x, y]$ of degree m.

Take $m = td + d - 1$ and consider the collection of $m + 1 = (t + 1)d$ polynomials

$$\{x^i y^j F_1(x, y)^k F_2(x, y)^\ell : i + j = d - 1, \ k + \ell = t\} \ \subset \ \Gamma_v^0(m) \ .$$

Let $\mathrm{Det}(m)$ denote the determinant of the matrix expressing these polynomials in terms of the standard monomials $\{x^m, x^{m-1}y, \ldots, y^m\}$.

Lemma 6.5. *For $m = td + d - 1$, $\mathrm{Det}(m) = \pm\mathrm{Res}(F)^{t(t+1)/2}$.*

Proof: We will first show that $\mathrm{Det}(m)$ vanishes if and only if $\mathrm{Res}(F)$ vanishes. Indeed, $\mathrm{Det}(m) = 0$ if and only if there is a nontrivial relation of the form

$$\sum_{i=0}^{t} h_i(x, y) F_1(x, y)^{t-i} F_2(x, y)^i \ = \ 0 \tag{6.6}$$

where each $h_i(x, y)$ is homogeneous of degree $d - 1$.

If (6.6) holds, let I be the least index for which $h_I(x, y) \neq 0$; necessarily $I < t$. Then $F_2(x, y)$ divides $h_I(x, y) F_1(x, y)^{t-I}$. Since $F_2(x, y)$ has degree d, it must have an irreducible factor in common with $F_1(x, y)$, so $\mathrm{Res}(F) = 0$. Conversely, if $\mathrm{Res}(F) = 0$ then there is a nontrivial relation

$$h_0(x, y) F_1(x, y) + h_1(x, y) F_2(x, y) \ = 0$$

where h_0 and h_1 are homogeneous of degree $d - 1$. Multiplying through by $F_1(x, y)^{t-1}$ gives a relation of the form (6.7).

Expand $\mathrm{Det}(m)$ and $\mathrm{Res}(F)$ as polynomials in the coefficients of F_1 and F_2. Comparing degrees and using the fact that $\mathrm{Res}(F)$ is irreducible, we see that

$$\mathrm{Det}(m) \ = \ C \cdot \mathrm{Res}(F)^{t(t+1)/2}$$

for some constant C. Taking $F_1 = x^d$, $F_2 = y^d$ and evaluating both sides, we find that $C = 1$ for an appropriate ordering of the terms. $\qquad\square$

We now turn to the proof of Proposition 6.4.

Proof:

We give the proof only when $R_v = 1$. The general case reduces to this by a scaling argument.

It suffices to prove the upper bound $S_\gamma(F^{-1}(B_v(1)), H) \leq |\mathrm{Res}(F)|_v^{-1/d^2}$ for each v. If this is known, then by the global equalities

$$\prod_v S_\gamma(F^{-1}(B_v(1)), H) \ = \ 1 \ ,$$

$$\prod_v |\mathrm{Res}(F)|_v \ = 1 \ ,$$

the local inequality must actually be an equality, for each v.

Write $E_v = F^{-1}(B_v(1))$. Since E_v is bounded, there is a constant $c_v \in \mathbb{C}_v$ such that $\|c_v x^i y^j\|_{E_v} \leq 1$ for all i, j with $i + j \leq 2d - 1$.

We will study $\mathrm{vol}_v(\mathcal{F}_v(n))$ by making use of the decomposition $\Gamma_v(n) = \oplus_{m=0}^n \Gamma_v^0(m)$, which is compatible with the monomial bases. For each $m \geq 0$, put

$$\mathcal{F}_v^0(m) \ = \ \{f \in \Gamma_v^0(m) : \|f\|_{E_v} \leq 1\} \ .$$

If $m \geq 2d - 1$, we can uniquely write $m = td + (d-1) + r$ with integers $t \geq 1$, $0 \leq r \leq d - 1$, and then

$$\Gamma_v^0(m) \;=\; y^r \cdot \Gamma_v^0(m - r) \oplus \Big(\bigoplus_{i=0}^{r-1} k_v \cdot x^{m-i} y^i\Big)$$

By Lemma 6.5, the polynomials $x^i y^j F_1^k F_2^\ell$ with $i + j = d - 1$, $k + \ell = t$ form a basis for $\Gamma_v^0(m - r)$, so the corresponding polynomials $x^i y^{j+r} F_1^k F_2^\ell$, together with the monomials $x^{m-i} y^i$ for $0 \leq i < r$, form a basis for $\Gamma_v^0(m)$. Again by Lemma 6.5, the transition matrix from the monomial basis for $\Gamma_v^0(m)$ to this new basis has determinant $\pm \mathrm{Res}(F)^{t(t+1)/2}$.

For each basis element of the first type, we have $i + j + r \leq 2d - 1$, so

$$c_v \cdot x^i y^{j+r} F_1^k F_2^\ell \;\in\; \mathcal{F}_v^0(m) \ .$$

For each basis element of the second type, $x^{m-i} y^i = x^{td} \cdot x^{d-1+r-i} y^i$ with $(d-1+r-i) + i \leq 2d - 1$, so

$$c_v^{t+1} \cdot x^{m-i} y^i \;\in\; \mathcal{F}_v^0(m) \ .$$

Now suppose v is nonarchimedean. By the ultrametric inequality, we have

$$\Big(\bigoplus_{\substack{i+j=d-1 \\ k+\ell=t}} \mathcal{O}_v \cdot c_v x^i y^{j+r} F_1^k F_2^\ell \Big) \oplus \Big(\bigoplus_{0 \leq i < r} \mathcal{O}_v \cdot c_v^{t+1} x^{m-i} y^i \Big) \subset \mathcal{F}_v^0(m) \tag{6.7}$$

and it follows that

$$\mathrm{vol}_v(\mathcal{F}_v^0(m)) \;\geq\; (|c_v|_v)^{(m-r+1)+r(t+1)} \cdot |\mathrm{Res}(F)|_v^{t(t+1)/2} \ .$$

Noting that $m/d > t > (m/d) - 1$, we see that

$$\log(\mathrm{vol}_v(\mathcal{F}_v^0(m))) \;\geq\; \frac{m^2}{2d^2} \log(|\mathrm{Res}(F)|_v) - O(m). \tag{6.8}$$

By increasing the implied constant, we can assume this holds for $m \leq 2d - 1$ as well.

For each $n \geq 0$, the ultrametric inequality shows that

$$\bigoplus_{m=0}^{n} \mathcal{F}_v^0(m) \;\subset\; \mathcal{F}_v(n) \ . \tag{6.9}$$

Using (6.8) and (6.9), it follows that

$$\begin{aligned}
-\log(S_\gamma(E_v, H)) \;&=\; \lim_{n \to \infty} \frac{3!}{n^3} \log(\mathrm{vol}_v(\mathcal{F}_v(n))) \\
&\geq\; \lim_{n \to \infty} \frac{3!}{n^3} \sum_{m=0}^{n} \Big(\frac{m^2}{2} \cdot \frac{1}{d^2} \log(|\mathrm{Res}(F)|_v) - O(m) \Big) \\
&=\; \frac{1}{d^2} \log(|\mathrm{Res}(F)|_v) \ .
\end{aligned} \tag{6.10}$$

Thus $S_\gamma(E_v, H) \leq |\mathrm{Res}(F)|_v^{-1/d^2}$, as desired.

If v is archimedean and $k_v \cong \mathbb{R}$, the triangle inequality gives a weaker containment; it is better to formulate the result directly for $\mathcal{F}_v(n)$. Noting that $\dim_k(\Gamma(n)) = (n+1)(n+2)/2$,

we obtain

$$\Big(\bigoplus_{i+j<2d-1} \tfrac{2}{(n+1)(n+2)} \cdot [-1,1] \cdot c_v x^i y^j \Big) \oplus$$

$$\bigoplus_{m=2d-1}^{n} \Bigg[\Big(\bigoplus_{\substack{m=td+d-1+r \\ k+\ell=t \\ i+j=d-1}} \tfrac{2}{(n+1)(n+2)} \cdot [-1,1] \cdot c_v x^i y^{j+r} F_1^k F_2^\ell \Big)$$

$$\oplus \Big(\bigoplus_{0 \le i < r} \tfrac{2}{(n+1)(n+2)} \cdot [-1,1] \cdot c_v^{t+1} x^{m-i} y^i \Big) \Bigg] \subset \mathcal{F}_v(n) .$$

From this inclusion, we deduce that

$$\log(\mathrm{vol}_v(\mathcal{F}_v(n))) \ \geq \ \frac{n^3}{6d^2} \log(|\mathrm{Res}(F)|_v) - O(n^2 \log(n)) . \tag{6.11}$$

If v is archimedean and $k_v \cong \mathbb{C}$ there is a similar containment, with $B(0,1)$ replacing $[-1,1]$. In either case, a computation like the one in (6.10) now shows that $S_\gamma(E_v, H) \leq |\mathrm{Res}(F)|_v^{-1/d^2}$. $\qquad\square$

6.2. **The sectional capacity of the filled Julia set.** Let the numbers r_v, R_v be as in Corollary 3.4. Put $K_F = \prod_v K_{F,v}$, $\mathbb{B}(\vec{r}) = \prod_v B_v(r_v)$, $\mathbb{B}(\vec{R}) = \prod_v B_v(R_v)$. Then for each n,

$$(F^{(n)})^{-1}(B_v(r_v)) \ \subseteq \ K_{F,v} \ \subseteq \ (F^{(n)})^{-1}(B_v(R_v)) , \tag{6.12}$$

$$(F^{(n)})^{-1}(\mathbb{B}(\vec{r})) \ \subseteq \ K_F \ \subseteq \ (F^{(n)})^{-1}(\mathbb{B}(\vec{R})) \ . \tag{6.13}$$

Theorem 6.14.

A) *The global sectional capacity $S_\gamma(K_F, H) = 1$. Equivalently,*

$$\prod_v S_\gamma(K_{F,v}, H) \ = \ 1 .$$

B) *For each v, $S_\gamma(K_{F,v}, H) = |\mathrm{Res}(F)|_v^{-1/(d(d-1))}$.*

Proof: Part (A) follows from Proposition 6.2 (C), using (6.13) and the fact that $F^{(n)}$ is homogeneous of degree d^n.

Part (B) comes out as follows. By [DeM, Corollary 6.4],

$$\mathrm{Res}(F^{(n)}) \ = \ \mathrm{Res}(F)^{(d^{2n-1}-d^{n-1})/(d-1)} .$$

Hence Proposition 6.4, applied to both halves of (6.12), gives

$$
\begin{aligned}
S_\gamma(K_{F,v}, H) \ &= \ \lim_{n \to \infty} |\mathrm{Res}(F^{(n)})|_v^{-1/(d^n)^2} \\
&= \ \lim_{n \to \infty} |\mathrm{Res}(F)|_v^{-(d^{2n-1}-d^{n-1})/d^{2n}(d-1)} \\
&= \ |\mathrm{Res}(F)|_v^{-1/d(d-1)} .
\end{aligned}
$$

$\qquad\square$

6.3. **The homogeneous sectional capacity.** Identify $\Gamma(n)$ with the space of polynomials in $k[x, y]$ of total degree $\leq n$. Instead of considering the asymptotics for volumes related to $\Gamma(n)$ in the definition of the sectional capacity, one can can consider the corresponding asymptotics for volumes concerning homogeneous polynomials alone. This gives rise to the homogeneous sectional capacity.

Put

$$\Gamma^0(n) \;=\; \{f(x, y) \in k[x, y] : f \text{ is homogeneous of degree } n\}$$

and write $\Gamma^0_v(n) = k_v \otimes_k \Gamma^0(n)$, $\Gamma^0_{\mathbb{A}}(n) = \mathbb{A}_k \otimes_k \Gamma^0(n)$. Put

$$\mathcal{F}^0_v(n) \;=\; \left\{f \in \Gamma^0_v(n) : \|f\|_{E_v} \leq 1\right\} ,$$
$$\mathcal{F}^0_{\mathbb{A}}(n) \;=\; \prod_v \mathcal{F}^0_v(n) \cap \Gamma^0_{\mathbb{A}}(n) .$$

Equipping $\Gamma^0_v(n)$ and $\Gamma^0_{\mathbb{A}}(n)$ with the bases given by the monomials $x^k y^\ell$, and giving those bases the term order $\prec$ given by the lexicographic order with $x \prec y$, graded by the degree, we have a situation analogous to that in the definition of the sectional capacity. By transport of structure, the Haar measure vol_v on k_v induces a measure vol_v on each $\Gamma^0_v(n)$, and the Haar measure $\text{vol}_{\mathbb{A}} = \prod_v \text{vol}_v$ on $\mathbb{A}_k$ induces a measure on each $\Gamma^0_{\mathbb{A}}(n)$. Define the local homogeneous sectional capacity $S^0_\gamma(E_v, H)$ by

$$-\log(S^0_\gamma(E_v, H)) \;=\; \lim_{n \to \infty} \frac{2!}{n^2} \log(\text{vol}_v(\mathcal{F}^0_v(n))) \tag{6.15}$$

and the global homogeneous sectional capacity $S^0_\gamma(\mathbb{E}, H)$ by

$$-\log(S^0_\gamma(E_v, H)) \;=\; \lim_{n \to \infty} \frac{2!}{n^2} \log(\text{vol}_{\mathbb{A}}(\mathcal{F}^0_{\mathbb{A}}(n))) . \tag{6.16}$$

The existence of these limits follows from the general existence theorem for sectional capacities of line bundles with "adelically normed sections" (see [RLV], Theorem A, p.4), applied to the variety $\mathbb{P}^1$ rather than $\mathbb{P}^2$. The details are as follows.

One can interpret the set of homogeneous polynomials $\Gamma^0(n)$ as the space of global sections $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$. The sup norms $\|f\|_{E_v}$ on the spaces $\Gamma^0_v(n)$ satisfy axioms (A1) and (A2) of ([RLV], p.3), because they are implied by Standard Hypotheses for sets ([RLV], Example 1.1, p.13). Then ([RLV], Theorem 6.2, p.66) asserts that the limit (6.15) exists, and ([RLV], Theorem 7.1, p.73) tells us that the limit (6.16) exists, and also that

$$S^0_\gamma(\mathbb{E}, H) \;=\; \prod_v S^0_\gamma(E_v, H) .$$

For each place v (archimedean or nonarchimedean), given a set $E_v \subset \mathbb{C}^2_v$, the *circled* set obtained from E_v is

$$E^0_v \;=\; \{wz : z \in E_v,\ w \in \mathbb{C}_v, |w|_v = 1\} .$$

We will call E_v *circled* if $E_v = E^0_v$. Trivially $S^0_\gamma(E^0_v, H) = S^0_\gamma(E_v, H)$.

Proposition 6.17. *For each bounded, Galois-stable set $E_v \subset \mathbb{C}^2_v$, we have $S_\gamma(E_v, H) \leq S^0_\gamma(E_v, H)$. If $E_v = E^0_v$, then $S_\gamma(E_v, H) = S^0_\gamma(E_v, H)$.*

Proof:

We first show that $S_\gamma(E_v, H) \leq S^0_\gamma(E_v, H)$. The proof breaks into two cases, according as $S^0_\gamma(E_v, H) > 0$ or $S^0_\gamma(E_v, H) = 0$.

First suppose $S_\gamma^0(E_v, H) > 0$, and fix $\varepsilon > 0$. By the definition of $S_\gamma^0(E_v, H)$, for all sufficiently large k

$$-\log(S_\gamma^0(E_v, H)) - \varepsilon \ \leq \ \frac{2!}{k^2} \log(\text{vol}_v(F_v^0(n))) \ \leq \ -\log(S_\gamma^0(E_v, H)) + \varepsilon \ . \tag{6.18}$$

Suppose v is nonarchimedean. By the ultrametric inequality, for each n we have

$$\bigoplus_{k=0}^n \mathcal{F}_v^0(k) \ \subset \ \mathcal{F}_v(n) \ .$$

Since the union of the monomial bases for the $\Gamma_v^0(k)$ is the monomial basis for $\Gamma_v(n)$, we have $\text{vol}_v(\mathcal{F}_v(n)) \geq \prod_{k=0}^n \text{vol}_v(\mathcal{F}_v(n))$. Hence

$$\begin{aligned}
-\log(S_\gamma(E_v, H)) \ &= \ \lim_{n \to \infty} \frac{3!}{n^3} \log(\text{vol}_v(\mathcal{F}_v(n)) \\
&\geq \ \lim_{n \to \infty} \frac{3!}{n^3} \sum_{k=0}^n \log(\text{vol}_v(\mathcal{F}_v(k)) \\
&\geq \ \lim_{n \to \infty} \frac{3!}{n^3} \sum_{k=0}^n \frac{k^2}{2!}(-\log(S_\gamma^0(E_v, H)) - \varepsilon) \\
&= \ -\log(S_\gamma^0(E_v, H)) - \varepsilon \ .
\end{aligned} \tag{6.19}$$

Since $\varepsilon > 0$ is arbitrary, $S_\gamma(E_v, H) \leq S_\gamma^0(E_v, H)$.

If v is archimedean, then $1/(n+1) \oplus_{k=0}^n \mathcal{F}_v^0(k) \ \subset \ \mathcal{F}_v(n)$ by the triangle inequality, so

$$\frac{3!}{n^3} \log \text{vol}_v(\mathcal{F}_v(n) \ \geq \ \frac{3!}{n^3} \sum_{k=0}^n \log \text{vol}_v(\mathcal{F}_v(k) - \frac{3!}{n^3} \cdot \frac{n(n+1)}{2} \log |n+1|_v \ .$$

Thus the factor of $1/(n+1)$ washes out in the asymptotics as $n \to \infty$, and the same computation as in (6.19) carries through.

If $S_\gamma^0(E_v, H) = 0$, take $M > 0$; then for all sufficiently large k,

$$\frac{2!}{k^2} \log(\text{vol}_v(F_v^0(n))) \ \geq \ M \ . \tag{6.20}$$

If v is nonarchimedean, using (6.20) in place of (6.18) in (6.19) shows $-\log(S_\gamma(E_v, H)) \geq M$. Since M is arbitrary, $S_\gamma(E_v, H) = 0$. A similar argument, with minor modifications to deal with the triangle inequality, applies in the archimedean case.

Now suppose $E_v = E_v^0$. We claim that $S_\gamma^0(E_v, H) \leq S_\gamma(E_v, H)$.

We will give the proof under the assumption that $S_\gamma^0(E_v, H) > 0$, leaving the other case to the reader.

Fix n, and suppose $f \in \mathcal{F}_v(n)$. Thus f is a polynomial in $k_v[x, y]$ of total degree at most n, with $\|f\|_{E_v} \leq 1$. Decompose $f = \sum_k f_k$ as the sum of its homogeneous parts of degree k. The f_k can be recovered from f by finite Fourier analysis: if ζ is a primitive $(n+1)$-st root of unity, then

$$f_k(z) \ = \ \frac{1}{n+1} \sum_{\ell=0}^n \zeta^{-\ell k} f(\zeta^\ell z) \ .$$

Since E_v is circled and $\|f\|_{E_v} \leq 1$, for each ℓ we have $\|f(\zeta^\ell z)\|_v \leq 1$.

If v is archimedean, then $\|f_k\|_{E_v} \leq 1$, and $\mathcal{F}_v(n) \subset \oplus_{k=0}^n \mathcal{F}_v^0(k)$. If v is nonarchimedean, then $(n+1)\mathcal{F}_v(n) \subset \oplus_{k=0}^n \mathcal{F}_v^0(k)$. The factor $n+1$ has no effect on the asymptotics of the volumes; in both the archimedean and nonarchimedean case, for each $\varepsilon > 0$ we have

$$
\begin{aligned}
-\log(S_\gamma(E_v, H)) &= \lim_{n\to\infty} \frac{3!}{n^3} \log(\mathrm{vol}_v(\mathcal{F}_v(n))) \\
&\leq \lim_{n\to\infty} \frac{3!}{n^3} \sum_{k=0}^n \frac{k^2}{2!}(-\log(S_\gamma^0(E_v, H)) + \varepsilon) \\
&= -\log(S_\gamma^0(E_v, H)) + \varepsilon \ ,
\end{aligned}
$$

which yields $S_\gamma^0(E_v, H) \leq S_\gamma(E_v, H)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 6.21. It can happen that $S_\gamma(E_v, H) < S_\gamma^0(E_v, H)$. For example, let $k = \mathbb{Q}$ and let v be the archimedean place. Take $E_v = [0,1] \times B(0,1) \subset \mathbb{C}^2$; then $E_v^0 = B(0,1) \times B(0,1)$. It is well known from the classical theory of logarithmic capacities that $\gamma_\infty([0,1]) = 1/4$ and $\gamma_\infty(B(0,1)) = 1$. By formula (6.3),

$$
S_\gamma(E_v, H) = 1/4 \ , \qquad S_\gamma(E_v^0, H) = 1 \ .
$$

But then Proposition 6.17 and the remarks preceding it imply that $S_\gamma^0(E_v, H) = S_\gamma^0(E_v^0, H) = S_\gamma(E_v^0, H) = 1$. Similar examples can be given for nonarchimedean v.

6.4. The homogeneous Chebyshev constant.

As before, let $E_v \subset \mathbb{C}_v^2$ be bounded and stable under $\mathrm{Gal}^c(\mathbb{C}_v/k_v)$. It will be useful to introduce another quantity equal to the homogeneous sectional capacity: the homogeneous Chebyshev constant. Its chief virtue is that it is independent of the ground field used to compute it.

For each field L with $k_v \subset L \subset \mathbb{C}_v$, put $\Gamma_L^0(n) = L \otimes_{k_v} \Gamma_v^0(n)$. (The cases of greatest interest are $L = k_v$ and $L = \mathbb{C}_v$.) The basis elements for $\Gamma^0(n)$ are $x^n \prec x^{n-1}y \prec \cdots \prec y^n$. For each $k = 0, \ldots, n$, define the set of "monic L-rational homogeneous polynomials of degree n with leading term $x^{n-k}y^k$" to be

$$
\Gamma_L^0(n, k) \ := \ \{f \in L \otimes_k \Gamma^0(n) : f(x,y) = x^{n-k}y^k + \sum_{i<k} a_i x^{n-i}y^i\}
$$

and put

$$
M_L(n, k) \ = \ \inf_{f \in \Gamma_L^0(n,k)} (\|f\|_{E_v})^{1/n} \ .
$$

By ([RLV], Theorem 6.1, p.64), the number $CH_L^0(E_v, H)$ defined by

$$
\log(CH_L^0(E_v, H)) \ = \ \lim_{n\to\infty} \frac{2!}{n}\left(\sum_{k=0}^n \log(M_L(n, k))\right)
$$

exists, and is independent of L. We will call it the homogeneous Chebyshev constant. By ([RLV], Theorem 6.2, p.66),

$$
S_\gamma^0(E_v, H) \ = \ CH_{k_v}^0(E_v, H) \ = \ CH_{\mathbb{C}_v}^0(E_v, H) \ . \tag{6.22}
$$

6.5. **The homogeneous transfinite diameter.** We have already introduced the homogeneous transfinite diameter $d_\infty^0(E_v)$. The definition makes sense for an arbitrary set $E_v \subset \mathbb{C}_v^2$. Under the assumptions that E_v is bounded and stable under $\mathrm{Gal}^c(\mathbb{C}_v/k_v)$, we will show that it coincides with the homogeneous sectional capacity.

Note that since the kernel $|z_i \wedge z_j|_v$ is invariant when z_i or z_j is multiplied by $w \in \mathbb{C}_v$ with $|w|_v = 1$, we clearly have $d_\infty^0(E_v) = d_\infty^0(E_v^0)$.

Proposition 6.23. *Let E_v be bounded and stable under* $\mathrm{Gal}^c(\mathbb{C}_v/k_v)$. *Then*

$$S_\gamma^0(E_v, H) \;=\; d_\infty^0(E_v) \; .$$

Proof: We claim that we can assume without loss that each $d_n(E_v) > 0$ and that each $\mathcal{F}_v(n) \subset \Gamma_v^0(n)$ is bounded.

If $d_n(E_v) = 0$ for some n, then there would be a finite set of points $\xi_1, \ldots, \xi_n$ with

$$E_v \;\subset\; \{w\xi_i : 1 \le i \le n, \; w \in \mathbb{C}_v\} \; .$$

This means $d_m^0(E_v) = 0$ for each $m > n$, so $d_\infty^0(E_v) = 0$. Also, for each $m > n$ the polynomial $f_m(z) = (\prod_{i=1}^n (z \wedge \xi_i)) \wedge (z \wedge \xi_1)^{m-n} \in \Gamma_m^0(n)$ vanishes on E_v, which means that $\mathrm{vol}_v(\mathcal{F}_v(m)) = \infty$ so $S_\gamma^0(E_v, H) = 0$.

On the other hand, if $\mathcal{F}_v(n)$ is unbounded for some n, then (by the local compactness of k_v) there would be a nonzero polynomial $f(z) \in \Gamma_v^0(n)$ with $\|f\|_{E_v} = 0$. Factoring $f(z) = \prod_{i=1}^n (z \wedge \xi_i)$, we find that again

$$E_v \;\subset\; \{w\xi_i : 1 \le i \le n, \; w \in \mathbb{C}_v\} \; ,$$

so $d_\infty^0(E_v) = S_\gamma^0(E_v, H) = 0$.

Henceforth we will assume that each $d_n^0(E_v) > 0$ and each $\mathcal{F}_v(n)$ is bounded.

We will first show that $d_\infty^0(E_v) \le S_\gamma^0(E_v)$.

Fix n. Given $\varepsilon > 0$, choose $\xi_1, \ldots, \xi_{n+1} \in E_v$ so that

$$|P_{n+1}(\xi_1, \ldots, \xi_{n+1})|_v \;\ge\; (d_{n+1}^0(E_v) - \varepsilon)^{(n+1)n/2} \; .$$

Write $\xi_\ell = (x_\ell, y_\ell)$ and let D be the $(n+1) \times (n+1)$ matrix whose ℓ-th column is given by the $x_\ell^{n-i} y_\ell^i$, $i = 0, \ldots, n$. Then $\det(D) = \pm \prod_{i<j} (\xi_i \wedge \xi_j)$. (To see this, first suppose each x_ℓ is nonzero. If x_ℓ^n is factored out from each column, we obtain a Vandermonde determinant in the y_ℓ/x_ℓ. When the x_ℓ^n are multiplied through in the formula for the Vandermonde, the formula for $\det(D)$ results. The general case follows by continuity.) Thus $|\det(D)|_v \ge (d_{n+1}^0(E_v) - \varepsilon)^{(n+1)n/2}$.

If v is nonarchimedean, the bounded $\mathcal{O}_v$-module $\mathcal{F}_v(n)$ has an $\mathcal{O}_v$-basis $\{g_0, \ldots, g_n\}$. Write $g_\ell(z) = \sum_{i=0}^n c_{\ell,i} x^i y^{n-i}$ for $i = 0, \ldots, n$, and let C be the $(n+1) \times (n+1)$ matrix with rows $c_{\ell,i}$, $0 \le i \le n$. Then $\mathrm{vol}_v(\mathcal{F}_v(n)) = |\det(C)|_v$.

For each $g \in \mathcal{F}_v(n)$ we have $|g(\xi_i)|_v \le 1$. Identifying a polynomial with its vector of coefficients, this says that $\mathcal{F}_v(n) \subset D^{-1} \widehat{\mathcal{O}}_v^{n+1}$. Passing to the the $\widehat{\mathcal{O}}_v$-module $\widehat{\mathcal{O}}_v \otimes_{\mathcal{O}_v} \mathcal{F}_v(n) = C\widehat{\mathcal{O}}_v^{n+1}$ generated by $\mathcal{F}_v(n)$, we conclude that $C\widehat{\mathcal{O}}_v^{n+1} \subset D^{-1}\widehat{\mathcal{O}}_v^{n+1}$, so $|\det(C)|_v \le |\det(D^{-1})|_v$. Thus

$$\mathrm{vol}_v(\mathcal{F}_v(n)) \;\le\; (d_{n+1}^0(E_v) - \varepsilon)^{-(n+1)n/2} \; ,$$

and hence

$$-\log(S_\gamma^0(E_v, H)) \;=\; \lim_{n\to\infty} \frac{2!}{n^2} \log(\mathrm{vol}_v(\mathcal{F}_v(n))) \tag{6.24}$$

$$\leq \lim_{n\to\infty} -\frac{n+1}{n} \log(d_{n+1}^0(E_v) - \varepsilon) \;=\; -\log(d_\infty^0(E_v) - \varepsilon) \;.$$

Letting $\varepsilon \to 0$, we obtain $d_\infty^0(E_v) \leq S_\gamma^0(E_v, H)$.

If v is archimedean, then by the same argument as above we find

$$\mathcal{F}_v(n) \;\subset\; D^{-1} B(0,1)^{n+1} \;.$$

If $k_v \cong \mathbb{C}$, this gives $\mathrm{vol}_v(\mathcal{F}_v(n)) \leq |\det(D)|_v^{-1} \cdot \pi^{n+1}$ (normalizing $|\ |_v$ so that $|w|_v = |w|^2$ for $w \in \mathbb{C}$). If $k_v \cong \mathbb{R}$, then the triangle inequality shows that

$$\frac{1}{2}(\mathcal{F}_v(n) \oplus i \cdot \mathcal{F}_v(n)) \;\subset\; D^{-1} B(0,1)^{n+1} \;,$$

which gives $\mathrm{vol}_v(\mathcal{F}_v(n)) \leq |\det(D)|_v^{-1} \cdot (2\sqrt{\pi})^{n+1}$. In either case a computation similar to that in (6.24) shows that $d_\infty^0(E_v) \leq S_\gamma^0(E_v, H)$.

Next we will show that $S_\gamma^0(E_v, H) \leq d_\infty^0(E_v)$.

Fixing n, we can choose $\xi_0, \ldots, \xi_n \in E_v$ such that

$$\frac{1}{2} d_{n+1}^0(E_v)^{(n+1)n/2} \;\leq\; |P_{n+1}(\xi_0, \ldots, \xi_n)|_v \;\leq\; d_{n+1}^0(E_v)^{(n+1)n/2} \;. \tag{6.25}$$

For each $\ell = 0, \ldots, n$, put

$$f_\ell(z) \;=\; \frac{\prod_{i\neq\ell}(z \wedge \xi_i)}{\prod_{i\neq\ell}(\xi_\ell \wedge \xi_i)} \;=\; \pm \frac{P_{n+1}(\xi_0, \ldots, \xi_{\ell-1}, z, \xi_{\ell+1}, \ldots, \xi_n)}{P_{n+1}(\xi_1, \ldots, \xi_{n+1})} \;. \tag{6.26}$$

Writing $z = (x, y)$, we can also expand $f_\ell(z)$ as a polynomial

$$f_\ell(z) \;=\; \sum_{i=0}^{n} a_{\ell,i} x^{n-i} y^i \;\in \mathbb{C}_v[x, y] \;.$$

By the definition of $d_{n+1}^0(E_v)$,

$$\sup_{z\in E_v} |P_{n+1}(\xi_0, \ldots, \xi_{\ell-1}, z, \xi_{\ell+1}, \ldots, \xi_n)|_v \;\leq\; d_{n+1}^0(E_v)^{(n+1)n/2} \;. \tag{6.27}$$

Combining (6.25), (6.26), and (6.27) shows that $\|f_\ell\|_{E_v} \leq 2$.

If v is nonarchimedean, let $\mathcal{G}_v(n) \subset \mathbb{C}_v \otimes_{k_v} \Gamma_v^0(n)$ be the $\widehat{\bar{\mathcal{O}}}_v$-module generated by $f_0(z), \ldots, f_n(z)$. Let A be the $(n+1) \times (n+1)$ matrix with entries $a_{\ell,i}$, and let D be the $(n+1) \times (n+1)$ matrix defined above, with columns obtained from $\xi_1, \ldots, \xi_{n+1}$. Since $f_\ell(\xi_i) = \delta_{\ell,i}$ it follows that $A \cdot D = I$. Hence

$$|\det(A)|_v \;=\; |\det(D)|_v^{-1} \;\geq\; d_{n+1}(E_v)^{-(n+1)n/2} \;. \tag{6.28}$$

The nonsingularity of A means that the $f_\ell(z)$ span $\mathbb{C}_v \otimes \Gamma_v^0(n)$. Using elementary row operations, we can transform A to a lower triangular matrix B with $\det(B) = \det(A)$. Equivalently, we can find a new $\widehat{\mathcal{O}}_v$-basis $\{g_0(z), \ldots, g_n(z)\}$ for $\mathcal{G}_v(n)$ having the form

$$g_\ell(z) \;=\; \sum_{i=0}^{\ell} b_{\ell,i} x^{n-i} y^i \;.$$

By the ultrametric inequality, $\|g_\ell\|_{E_v} \leq 2$ for each ℓ. Dividing through by $b_{\ell,\ell}$ we obtain a monic homogeneous polynomial

$$h_\ell(z) \;=\; x^{n-\ell}y^\ell + \sum_{i=0}^{\ell-1} \frac{b_{\ell,i}}{b_{\ell,\ell}} x^{n-i}y^i$$

with sup norm $\|h_\ell\|_{E_v} \leq 2|b_{\ell,\ell}|_v^{-1}$. From the definition of $M_{\mathbb{C}_v}(n,\ell)$, we obtain

$$M_{\mathbb{C}_v}(n,\ell)^n \;\leq\; 2|b_{\ell,\ell}|_v^{-1} . \tag{6.29}$$

Using (6.29) and (6.28), together with $|\det(B)|_v = |\det(A)|_v$,

$$
\begin{aligned}
\frac{2!}{n}\sum_{\ell=0}^{n} \log_v(M_{\mathbb{C}_v}(n,\ell)) &\leq \frac{2}{n^2}\sum_{\ell=0}^{n}\log(2/|b_{\ell,\ell}|_v) \\
&= \frac{2}{n^2}\log(2) - \frac{2}{n^2}\log(|\det(B)|_v) \\
&\leq \frac{2}{n^2}\log(2) + \frac{n+1}{n}\log(d_{n+1}^0(E_v)) .
\end{aligned}
$$

Passing to the limit as $n \to \infty$, we get

$$CH_{\mathbb{C}_v}^0(E_v) \;\leq\; d_\infty^0(E_v) .$$

Since $S_\gamma^0(E_v, H) = CH_{\mathbb{C}_v}^0(E_v)$, we are done.

If v is archimedean, fix n and let the $f_\ell(z) \in \mathbb{C}[x,y]$ be as above; each $f_\ell(z)$ satisfies $\|f_\ell\|_{E_v} \leq 2$. If $k_v \cong \mathbb{C}$, then by the triangle inequality

$$\frac{1}{n+1}\cdot\frac{1}{2}\cdot\bigoplus_{\ell=1}^{n} B(0,1)f_\ell \;\subset\; \mathcal{F}_v(n) .$$

Introducing the matrices A and D as before, and identifying a polynomial with its vector of coefficients, we have $1/(2(n+1))\cdot A\cdot B(0,1)^{n+1} \subset \mathcal{F}_v(n)$. If $k_v \cong \mathbb{R}$, then

$$\frac{1}{n+1}\cdot\frac{1}{2}\cdot\bigoplus_{\ell=1}^{n} B(0,1)f_\ell \;\subset\; \mathcal{F}_v(n)\oplus i\mathcal{F}_v(n) \;\subset\; \mathbb{C}\otimes_{k_v}\Gamma_v^0(n)$$

and $1/(2(n+1))\cdot A\cdot B(0,1)^{n+1} \subset \mathcal{F}_v(n)\oplus i\mathcal{F}_v(n)$.

In either case,

$$
\begin{aligned}
\mathrm{vol}_v(\mathcal{F}_v(n)) &\geq |2(n+1)|_v^{-(n+1)}\cdot|\det(A)|_v\cdot\pi^{n+1} \\
&\geq |2(n+1)|_v^{-(n+1)}\cdot\pi^{n+1}\cdot d_{n+1}^0(E_v)^{-(n+1)n/2} .
\end{aligned}
$$

Thus

$$
\begin{aligned}
-\log(S_\gamma^0(E_v, H)) &= \lim_{n\to\infty}\frac{2!}{n^2}\log(\mathrm{vol}_v(\mathcal{F}_v(n))) \\
&\geq \lim_{n\to\infty}\frac{n+1}{n}(-\log(d_{n+1}^0(E_v))) = -\log(d_\infty^0(E_v)),
\end{aligned}
$$

or equivalently, $S_\gamma^0(E_v, H) \leq d_\infty^0(E_v)$. $\qquad\square$

As we have noted before, the homogeneous transfinite diameter is defined for arbitrary sets $E_v \subset \mathbb{C}_v^2$. When $F = (F_1, F_2)$ is defined over a number field, then Propositions 6.17, 6.23, and Theorem 6.14 tell us that the filled Julia set $K_{F,v}$ satisfies

$$S_\gamma(K_{F,v}, H) \;=\; S_\gamma^0(K_{F,v}, H) \;=\; d_\infty^0(K_{F,v}) \;=\; |\mathrm{Res}(F)|_v^{-1/d(d-1)} . \tag{6.30}$$

In particular, we have finally proved Theorem 3.16.

Remark 6.31. Note that Corollary 3.18 follows directly from Propositions 6.17, 6.23 and Theorem 6.14(A), and in particular Proposition 6.4 is not needed for the proof.

Finally, we show that the last equality in (6.30) holds for arbitrary polynomial maps defined over $\mathbb{C}_v$ (and not just for maps defined over k).

Corollary 6.32. *Let $F_1, F_2 \in \mathbb{C}_v[x, y]$ be homogeneous polynomials of degree $d \geq 2$, having no common factor. Put $F = (F_1, F_2)$ and let $R_v > 0$. Then*
A) $d_\infty^0(F^{-1}(B_v(R_v))) = (R_v^2)^{1/d} \cdot |\mathrm{Res}(F)|_v^{-1/d^2}$,
B) $d_\infty^0(K_{F,v}) = |\mathrm{Res}(F)|_v^{-1/(d(d-1))}$.

Proof: Part (B) follows from (A) by the proof of Theorem 6.14, so it suffices to prove (A).

First suppose v is nonarchimedean. Since $\overline{\mathbb{Q}}$ is dense in $\mathbb{C}_v$, there are polynomials $\tilde{F}_1, \tilde{F}_2$ defined over a number field k such that the map $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ satisfies $\tilde{F}^{-1}(B_v(R_v)) = F^{-1}(B_v(R_v))$, and such that $|\mathrm{Res}(\tilde{F})|_v = |\mathrm{Res}(F)|_v$. Let w be the place of k induced by the given embedding $k \hookrightarrow \mathbb{C}_v$. The normalized absolute value $|\ |_w$ on $\mathbb{C}_w \cong \mathbb{C}_v$ is a power of $|\ |_v$, say $|\ |_w = |\ |_v^D$. For each set $E \subset \mathbb{C}_w \cong \mathbb{C}_v$, the homogeneous transfinite diameters $d_\infty^0(E)_w$ and $d_\infty^0(E)_v$ computed relative to $|\ |_w$ and $|\ |_v$, are related by the same power D. Hence our assertion follows from (6.30) for $\tilde{F}$.

If v is archimedean, choose a sequence $\{s_1, s_2, \dots\}$ decreasing monotonically to R_v from above, and another sequence $\{r_1, r_2, \dots\}$ increasing monotonically to R_v from below. By continuity, for each n we can choose $\overline{\mathbb{Q}}$-rational polynomials $\tilde{F}_{n,1}, \tilde{F}_{n,2}$ close enough to F_1 and F_2 that the maps $\tilde{F}_n = (\tilde{F}_{n,1}, \tilde{F}_{n_2})$ satisfy $\tilde{F}_n^{-1}(B_v(r_n)) \subset F^{-1}(B_v(R_v)) \subset \tilde{F}_n^{-1}(B_v(s_n))$ and

$$\lim_{n \to \infty} \mathrm{Res}(\tilde{F}_n) = \mathrm{Res}(F) .$$

If $K_v \cong \mathbb{R}$, we also require that the coefficients of the $\tilde{F}_{n,i}$ belong to $\mathbb{R}$. Applying (6.30) to the sets $\tilde{F}_n^{-1}(B_v(r_n))$, $\tilde{F}_n^{-1}(B_v(s_n))$ and taking a limit, we obtain (A). $\square$

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